**Theorem** If $A, B$ are positive semidefinite, then

$$\text{tr}(AB) \leq \text{tr}(A)\text{tr}(B).$$

**Proof.** First suppose $B$ is diagonal with eigenvalues $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$. It follows that

$$\text{tr}(AB) = \sum a_{ii}\lambda_i \leq \lambda_n \sum a_{ii} = \lambda_n \text{tr}(A) \leq \text{tr}(A)\text{tr}(B).$$

Here we used that $a_{ii} \geq 0$, which follows from the assumption that $A \succeq 0$.

In general, suppose $T$ is a unitary matrix such that $B = T^*DT$, where $D$ is diagonal with eigenvalues $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$. Then $TAT^* \succeq 0$ and $\text{tr}(TAT^*) = \text{tr}(A)$, $	ext{tr}(D) = \text{tr}(B)$. By the result just proved, we obtain

$$\text{tr}(AB) = \text{tr}(AT^*DT) = \text{tr}(TAT^*D) \leq \text{tr}(TAT^*)\text{tr}(D) = \text{tr}(A)\text{tr}(B).$$