General Algebraic and Differential
Riccati Equations from Stochastic LQR Problem

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Abstract. In this paper we consider a class of matrix Riccati equations arising from stochastic LQR problems. We prove a monotonicity of solutions to the differential Riccati equations, which leads to a necessary and sufficient condition for the existence of solutions to the algebraic Riccati equations. In addition, we obtain results on comparison, uniqueness, stabilizability and approximation for solutions of the algebraic Riccati equations.

§ 1. Introduction

The following differential Riccati equation has been studied in [12] by using the method of upper and lower solutions.

\[
P' + A^T P + PA + C^T PC + G + \Pi(P) - (B^T P + D^T PC + S)^T (R + D^T PD)^{-1} (B^T P + D^T PC + S) = 0,
\]

where \(A^T\) is the transpose of \(A\), \(P = \frac{dP}{dt}\), \(N\) is a symmetric matrix, \(\Pi\) is a linear map of symmetric matrices, and \(A, B, C, D, G, R\) and \(S\) are bounded and measurable matrix functions with appropriate dimensions. The main results in [12] include an interpretation of upper and lower solutions, comparison theorems, an upper-lower solution theorem, necessary and sufficient conditions for existence of solutions, an estimation of maximal existence intervals of solutions and an approximation of solutions.

This paper is a continuation of [12]. The focus here is the algebraic equation associated with (1):

\[
A^T P + PA + C^T PC + G + \Pi(P) - (B^T P + D^T PC + S)^T (R + D^T PD)^{-1} (B^T P + D^T PC + S) = 0,
\]

where \(A, B, C, D, G, R\) and \(S\) are constant matrices with appropriate dimensions; see (5) below.

The inclusion of the term \(\Pi\) is important to stochastic control problems with Markovian jumping noises and differential game problems with state-dependent noises; see [18] and [11], for example.

Equation (2) with \(\Pi = 0\) becomes

\[
A^T P + PA + C^T PC + G - (B^T P + D^T PC + S)^T (R + D^T PD)^{-1} (B^T P + D^T PC + S) = 0.
\]

Equation (3) arises from the following stochastic linear quadratic regulator (LQR) problem of infinite-
horizon with control-dependent noises:

\[
\begin{cases}
\text{minimize/maximize } J(u) \text{ for } u \in \mathcal{U}(0, \infty), \text{ where} \\
J(u) = E\{ \int_0^\infty (x^T G x + 2x^T Su + u^T Ru) dt \} \text{ subject to} \\
dx = (Ax + Bu) dt + (Cx + Du) dW, \ t \geq 0; \ x(0) = z,
\end{cases}
\]  

(4.1) \hspace{1cm} (4)

where \( W(t) \) is a standard Brownian motion on a complete probability space with \( W(0) = 0 \) almost surely, \( E\{\} \) is the expectation of the enclosed variable, and \( \mathcal{U}(0, \infty) \) is the set of all admissible control processes \( u \); see Section 2 for a further description. For a detailed account of this problem, see [1] and [23]. Through approaches of convex optimization and semidefinite programming, interesting relationships between equation (3) and the LQR problem (4) have established in [1] and [23]. These approaches are especially attractive for purpose of approximating solutions.

A monotonicity property for solutions to equation (1) will be proved, which leads to necessary and sufficient conditions for the existence of solutions to (2). For solutions to equation (2), we will prove results on comparison, uniqueness, stabilizability and approximation. We use a method of upper and lower solutions, which satisfy associated Riccati inequalities. This method is closely related to the methods of linear matrix inequalities in [1, 19, 23] and semidefinite programming in [23]. As pointed out in [12], our method has several desirable merits. For example, it directly links equation (3) with the LQR problem (4); see Theorem 2. It derives the main results under general assumptions. It gives verifiable necessary and sufficient conditions for the existence of solutions [Theorems 8 and 9]. It also gives algorithms for approximating solutions. It can be used to estimate the maximal existence intervals of solutions to differential Riccati equations without solving the equations [12]. It applies to differential as well as algebraic Riccati equations.

The paper is organized as follows. In Section 2, we set up the notations used in this paper and define upper and lower solutions. In addition, we describe the LQR problem (4) and interpret upper and lower solutions to (2) in Theorem 2.

Section 2 is ended with some results from [12] that are needed in this paper. Specifically, Theorem 5 is an upper-lower solution theorem and Propositions 3 and 4 are some structural properties of equations (1), (2) and (3).

In Section 3, we prove a monotonicity [Theorem 7] for solutions to (1), which leads to a general necessary and sufficient condition [Theorem 8] for the existence of solutions to (2). A refined existence result [Theorem 9] for (2) is proved under ms-stabilizability. Theorem 9 generalizes a main result in [1, Theorem 10] for (2) with \( \Pi = S = 0 \). The definitions of ms-stability, ms-stabilizability and ms-detectability for equation (2) are given in this section.

In Section 4, we study the comparison, uniqueness, stabilizability and approximation of solutions to (2). Theorem 11 is a comparison theorem for ms-stabilizing solutions. Theorem 12 gives the uniqueness and extreme properties of stabilizing solutions. Generalizing a classical relationship between detectability and stability, Theorem 14 shows that ms-detectability implies the ms-stabilizability of solutions to (2). Theorem 16 gives an algorithm for approximating solutions to (2). The results here generalize known results for equation (2) with \( C = D = S = 0 \) obtained in [6], [7] and [22]. For results for equation (2) with \( C = D = S = \Pi = 0 \), see [2], [3], [4], [5], [9], [10], [15], [16], [17], [19], [20], [21] and [25]. The proofs are reduced to proving the same results for linear equations in Theorems 10 and 13, which may be of interest in their own right.

§ 2. Preliminary Results
Denote $S^n = \{ P \in \mathbb{R}^{n \times n} : P^T = P \}$, where $P^T$ is the transpose of $P$. We write $M_1 \geq M_2$ ($M_1 > M_2$) if $M_1 - M_2 \in S^n$ is a positive semidefinite (definite). For a map $\Pi : S^n \rightarrow S^n$ we write $\Pi \geq 0$ if $\Pi(M_1) \geq 0$ for each $M_1 \geq 0$.

**Assumption.** We assume that $A, B, C, D, G, R, S$ and $N$ in equations (1), (2) and (3) are constant matrices and $\Pi : S^n \rightarrow S^n$ is a linear map, which satisfy

$$A, C \in \mathbb{R}^{n \times n}; \quad B, D, S^T \in \mathbb{R}^{n \times k}; \quad R \in \mathbb{S}^k; \quad G, N \in S^n; \quad \Pi \geq 0. \tag{5}$$

For a Hilbert space $\mathbb{X}$ and an interval $I$, $L^\infty(I, \mathbb{X})$ is the space of all bounded and measurable functions from $I$ to $\mathbb{X}$. Furthermore, we define $L^{1, \infty}(I, \mathbb{X}) = \{ P \in L^\infty(I, \mathbb{X}), P' \in L^\infty(I, \mathbb{X}) \}$. The solution $P$ to (1) is assumed to be in $L^{1, \infty}(I, S^n)$. Since all matrices in (1) are constant, a solution $P$ on $I$ is actually smooth. The solution $P$ to (2) and (3) is assumed to be in $S^n$, which may be considered as a constant solution to the associated differential equation.

As in [12], we abbreviate (1), (2) and (3) as

$$\begin{cases}
P' + LQ(P) + \Pi(P) = 0, \quad P(t_1) = N, \\
LQ(P) + \Pi(P) = 0, \\
LQ(P) = 0,
\end{cases} \tag{1} \tag{2} \tag{3}$$

where $LQ(P) = G + L(P) - Q(P)$ and

$$\begin{cases}
L(P) = A^T P + P A + C^T P C, \\
Q(P) = (B^T P + D^T PC + S)^T (R + D^T PD)^{-1} (B^T P + D^T PC + S)
\end{cases} \tag{6}$$

We remark that $Q(P)$ and $LQ(P)$ make sense even if $R + D^T PD$ is singular. To see this, we introduce the following notations

$$\mathcal{R}(P) = R + D^T PD, \quad \mathcal{S}(P) = B^T P + D^T PC + S. \tag{7}$$

$$\mathcal{K}(P) = \begin{cases}
\mathcal{R}(P)^{-1} \mathcal{S}(P) & \text{if } \mathcal{R}(P) \text{ is nonsingular}, \\
\mathcal{R}(P)^+ \mathcal{S}(P) & \text{if } \mathcal{R}(P) \text{ is singular},
\end{cases} \tag{8}$$

where $\mathcal{R}(P)^+$ is the pseudoinverse of $\mathcal{R}(P)$. Recall that any matrix $M$ has a unique Moore-Penrose pseudoinverse $M^+$ with the following properties (see [14] and [1]).

$$MM^+ M = M, \quad M^+ MM^+ = M^+, \tag{9}$$

If $M \in S^n$, then $M^+ \in S^n$, and $MM^+ = M^+ M$.

$$M \geq 0 \text{ if and only if } M^+ \geq 0. \tag{10}$$

Although $\mathcal{K}(P) = \mathcal{R}(P)^+ \mathcal{S}(P)$ always exists, but it may not be bounded nor satisfy

$$\mathcal{S}(P) = \mathcal{R}(P) \mathcal{K}(P) \tag{10}$$

on the interval $I$. We recall some definitions in [12]. A function $P \in L^{1, \infty}(I, S^n)$ is said to be feasible if $\mathcal{K}(P) \in L^\infty(I, \mathbb{R}^{k \times n})$ and (10) holds. A function $K \in L^\infty(I, \mathbb{R}^{k \times n})$ is called a feedback matrix associated with $P \in L^{1, \infty}(I, S^n)$ if it satisfies $\mathcal{S}(P) = \mathcal{R}(P) K$. The set of all such $K's$ is denoted by
Suppose $\mathcal{K}(P) \in L^\infty(I, \mathbb{R}^{k \times n})$, then $P$ is feasible if and only if $\mathbb{K}(P) \neq \emptyset$, and in this case,

$$Q(P) = \mathcal{K}(P)^T\mathcal{R}(P)\mathcal{K}(P) = K^T\mathcal{R}(P)K,$$

(11)

for each $K \in \mathbb{K}(P)$; see [12] for proof. In other words, $Q(P)$ is well-defined by (11) whenever $P$ is feasible.

If $P \in \mathbb{S}^n$, then $P$ is feasible as long as (10) holds. The term "feasible" is consist with that defined in [23] for the associated semidefinite programming problem.

**Definition 1.** $P \in L^1,\infty(I, \mathbb{S}^n)$ is a solution (upper solution, lower solution) to (1) if

$$LQ(P) + LQ(P) + \Pi(P) = 0 \quad (\leq 0, \geq 0), \quad P(t_1) = N \quad (\geq N, \leq N).$$

An upper or lower solution is strict if one of the inequalities is strict. Similarly, $P \in \mathbb{S}^n$ is a solution (upper, lower solution) to (2) if $LQ(P) + \Pi(P) = 0 \quad (\leq 0, \geq 0$, respectively).

Upper and lower solutions can be interpreted in terms of the well-definedness of the LQR Problem (4). Let $L^2(\mathbb{R}^k)$ (same as $L^2(\mathbb{R}^n)$) in [11] be space of $\mathbb{R}^k$-valued processes $u$ on $[0,\infty)$ that are adapted to the $\sigma$-field generated by $W(t)$ such that $E\int_0^\infty |u|^2 dt < \infty$ (square-integrable). Each $u \in L^2(\mathbb{R}^k)$ is an open-loop control. We say that $u \in L^2(\mathbb{R}^k)$ is ms-stabilizing if the solution $x$ to equation (4.2) satisfies $E\{ |x(t)|^2 \} \to 0$ as $t \to \infty$. The state equation (4.2) is stabilizable if there exists a feedback matrix $K \in \mathbb{R}^{k \times n}$ such that $u = -Kx$ is stabilizing, where $x$ is the solution to

$$dx = (A - BK)xdt + (C - DK)x dW, \quad x(0) = z,$$

(12)

which is induced from (4.2) by $u = -Kx$. In this case, $K$ is called a stabilizing feedback matrix. A solution $P \in \mathbb{S}^n$ to (3) is said to be stabilizing if $u = -Kx$ is stabilizing.

These concepts will be generalized for equation (2) without referring to the state equation (4.2).

Let $\mathcal{U}(0,\infty)$ be the set of all $u \in L^2(\mathbb{R}^k)$ such that the solution $x$ to equation (4.2) is $L^2$-integrable; that is, $E\{ \int_0^\infty |x(t)|^2 dt \} < \infty$. Clearly $J(u)$ is well-defined for each $u \in \mathcal{U}(0,\infty)$. Furthermore, the following relationship holds.

**Proposition 1.** Each $u \in \mathcal{U}(0,\infty)$ is stabilizing. Conversely, if $K \in \mathbb{R}^{k \times n}$ such that $u = -Kx$ is stabilizing, then $u \in \mathcal{U}(0,\infty)$.

Proof. If $u \in \mathcal{U}(0,\infty)$, then the solution $x$ to (4.2) satisfies $E\{ \int_0^\infty |x(t)|^2 dt \} < \infty$, which implies that $E\{ |x(t)|^2 \} \to 0$ as $t \to \infty$. In other words, $u$ is stabilizing. Conversely, if $u = -Kx$ is stabilizing, then by [1, Thm 1], there exists $P \in \mathbb{S}^n$, $P > 0$, such that $\mathcal{L}(K; P) < 0$, where

$$\mathcal{L}(K; P) = (A - BK)^T P + P(A - BK) + (C - DK)^T P(C - DK),$$

(13)

which is $L(P)$ with $A$ and $C$ replaced by $A - BK$ and $C - DK$, respectively. By Ito’s lemma and the fundamental theorem of calculus, we have

$$E\{ x^T(t_1)Px(t_1) \} = z^T P z + E \int_0^{t_1} \frac{d}{dt} x^T(t)Px(t) dt$$

$$= z^T P z + E \int_0^{t_1} x^T \mathcal{L}(K; P)x dt.$$  

(14)
From (14) and the fact that \( E\{ |x(t_1)|^2 \} \to 0 \) as \( t_1 \to \infty \), one has that 
\[
E \int_0^\infty x^T L(K; P)x \, dt = -z^T P z.
\]
Since \( L(K; P) < 0 \), it follows that \( E\{ \int_0^\infty |x(t)|^2 \, dt \} < \infty \). So \( u \in \mathcal{U}[0, \infty) \). □

**Theorem 2.** Suppose \( P \in \mathbb{S}^n \) is feasible and stabilizing.

(i) If \( LQ(P) \geq 0 \) and \( \mathcal{R}(P) \geq 0 \), then \( J(u) \geq z^T P z \) for \( u \in \mathcal{U}[0, \infty) \).

(ii) If \( LQ(P) \leq 0 \) and \( \mathcal{R}(P) \leq 0 \), then \( J(u) \leq z^T P z \) for \( u \in \mathcal{U}[0, \infty) \).

(iii) If \( LQ(P) = 0 \) and \( \mathcal{R}(P) \geq 0 \) (or \( \mathcal{R}(P) \leq 0 \)), then \( z^T P z \) is the minimum (maximum, respectively) value of \( J(u) \) over \( \mathcal{U}[0, \infty) \), which occurs when \( u = -Kx \), where \( K \in \mathbb{K}(P) \) and \( x \) satisfies (12).

**Proof.** Suppose \( u \in \mathcal{U}[0, \infty) \) and \( x \) is the solution to equation (4.2). By the Fundamental Theorem of calculus and Ito’s formula, we have
\[
E\{ x^T(t_1)P x(t_1) \} = z^T P z + E \int_0^{t_1} \frac{d}{dt} x^T(t)P x(t) \, dt
\]
\[
= z^T P z + E \int_0^{t_1} \{ x^T L(P) x + 2u^T (B^T P + D^T P C) x + u^T D^T P D u \} \, dt,
\]
where \( L(P) = A^T P + P A + C^T P C \) as defined in (6). Taking limits in (0), we obtain
\[
0 = z^T P z + E \int_0^\infty \{ x^T L(P) x + 2u^T (B^T P + D^T P C) x + u^T D^T P D u \} \, dt.
\]
Adding (0) to \( J(u) \) and using the notations \( \mathcal{R}(P) \) and \( \mathcal{S}(P) \) in (7), we obtain
\[
J(u) = z^T P z + E \int_0^\infty \{ x^T (L(P) + G)x + 2u^T \mathcal{S}(P) x + u^T \mathcal{R}(P) u \} \, dt.
\]
Since \( \mathcal{S}(P) = \mathcal{R}(P) K \) for each \( K \in \mathbb{K}(P) \), we can write
\[
2u^T \mathcal{S}(P) x + u^T \mathcal{R}(P) u = (u + Kx)^T \mathcal{R}(P) (u + Kx) - K^T \mathcal{R}(P) K.
\]
Recall that \( LQ(P) = G + L(P) - K^T \mathcal{R}(P) K \). It follows that
\[
J(u) = z^T P z + E \int_0^\infty \{ x^T LQ(P) x + (u + Kx)^T \mathcal{R}(P) (u + Kx) \} \, dt.
\]
In case (i), we have \( LQ(P) \geq 0 \) and \( \mathcal{R}(P) \geq 0 \), so (0) implies that \( J(u) \geq z^T P z \) for every \( u \in \mathcal{U}[0, \infty) \). Similarly, in case (ii), (0) implies that \( J(u) \leq z^T P z \) for every \( u \in \mathcal{U}[0, \infty) \). In case (iii), (0) implies that for every \( u \in \mathcal{U}[0, \infty) \),
\[
J(u) = z^T P z + E \int_0^\infty \{ (u + Kx)^T \mathcal{R}(P) (u + Kx) \} \, dt.
\]
It follows that \( J(u) \) has a minimum (maximum) \( z^T P z \) at \( u = -Kx \) if \( \mathcal{R}(P) \geq 0 \) (if \( \mathcal{R}(P) \leq 0 \)).

Equation (12) is precisely the state equation (4.2) with \( u = -Kx \). □

Note that when \( u = -Kx \), the cost \( J(u) \) becomes \( J_K = \int_0^\infty x^T \mathcal{G}(K)x \, dt \), where
\[
\mathcal{G}(K) = K^T R K - K^T S - S^T K + G.
\]
(15)
The following two propositions and Theorem 5 are proved in [12].

**Proposition 3.** Suppose $P \in L^{1,\infty}(I, S^n)$ is feasible and $K \in L^{\infty}(I, \mathbb{R}^{k \times n})$. Let $\mathcal{L}(P)$, $\mathcal{L}(K; P)$ and $\mathcal{G}(K)$ be defined in (6), (13) and (15), respectively. Then

(i) \[ \mathcal{L}(P) + (K(P) - K)^T \mathcal{R}(P)(K(P) - K) = \mathcal{G}(K) + \mathcal{L}(K; P), \] \hspace{1cm} (16)

(ii) \[ \begin{cases} \mathcal{L}(P) \leq \mathcal{G}(K) + \mathcal{L}(K; P), & \text{if } \mathcal{R}(P) \geq 0 \\ \mathcal{L}(P) \geq \mathcal{G}(K) + \mathcal{L}(K; P), & \text{if } \mathcal{R}(P) \leq 0 \end{cases} \] \hspace{1cm} (17)

**Proposition 4.** Suppose $Y, Z \in L^{1,\infty}(I, S^n)$ are feasible, $K \in L^{\infty}(I, \mathbb{R}^{k \times n})$. Denote $P = Y - Z$, $\widehat{A} = A - BK(Z)$, $\widehat{C} = C - DK(Z)$ and $\widehat{R} = \mathcal{R}(Z)$. Then

(i) \[ \mathcal{L}(Y) - \mathcal{L}(Z) \]
\[ = \widehat{A}^T P + P \widehat{A} + \widehat{C}^T P \widehat{C} - (B^T P + D^T P \widehat{C})^T (\widehat{R} + D^T P D)^+ (B^T P + D^T P \widehat{C}) \] \hspace{1cm} (18)

(ii) If $Z$ is given, then $Y$ satisfies equation (1) if and only if $P = Y - Z$ satisfies

\[ \begin{cases} P' + \widehat{G} + \widehat{A}^T P + P \widehat{A} + \widehat{C}^T P \widehat{C} + \Pi(P) \\ - (B^T P + D^T P \widehat{C})^T (\widehat{R} + D^T P D)^+ (B^T P + D^T P \widehat{C}) = 0, \end{cases} \] \hspace{1cm} (19)

where $\widehat{G} = Z' + LQ(Z) + \Pi(Z)$.

Note that $Z$ is a lower solution to (1) if and only if $\widehat{G} \geq 0$ and $N - Z(t_1) \geq 0$, which is equivalent to that $0$ is a lower solution to (19). Similarly, $Z$ is an upper solution to (1) if and only if $0$ is an upper solution to (19). In other words, equation (1) has an upper or lower solutions is equivalent to that (1) can be translated to a *standard* problem that has $0$ as an upper or lower solution.

Also note that Propositions 3 and 4 hold, in particular, for $P, Y, Z \in S^n$ and $K \in \mathbb{R}^{k \times n}$.

**Theorem 5** (upper-lower solution theorem). Suppose that $(Y, Z)$ is a pair of upper-lower solutions to (1).

(i) If either $\mathcal{R}(Z) \geq 0$ or $\mathcal{R}(Y) \leq 0$, then $Y \geq Z$. In addition, if one of $Y$ and $Z$ is strict, then $Y > Z$.

(ii) If either $\mathcal{R}(Z) > 0$ or $\mathcal{R}(Y) < 0$, then equation (1) has a unique solution $P$ with $Y \geq P \geq Z$.

**Remark 6.** As noted in [12], if a lower solution $Z$ (if it exists) to (1) with $\mathcal{R}(Z) > 0$ has certain property, then an upper solution $Y$ (if it exists) to (1) with $\mathcal{R}(Y) < 0$ also has the corresponding property. Same is true for equations (2) and (3). Because of this, we will state properties of upper solutions without proof.

§3. Monotonicity of Solutions to (1) and Existence of Solutions to (2)

Now we consider equations (1) and (2), which are
\[ \mathcal{E}(P) \equiv P' + \text{LQ}(P) + \Pi(P) = 0, \quad P(t_1) = N, \quad (1) \]

\[ \mathcal{E}(P) \equiv \text{LQ}(P) + \Pi(P) = 0. \quad (2) \]

By the local theory of differential equations, equation (1) has a unique solution in a maximum interval \((t_0, t_1]\). We show that this solution is monotone if and only if \(N\) is a lower or upper solution to (2); that is, \(\mathcal{E}(N) \geq 0\) or \(\mathcal{E}(N) \leq 0\).

**Theorem 7** (Monotonicity of Solutions). Suppose \(P\) is the feasible solution of (1) in \((t_0, t_1]\). Then we have

(i) \(\mathcal{E}(N) \geq 0\) if and only if \(P\) is increasing in \((t_0, t_1]\) as \(t\) decreases.

(ii) \(\mathcal{E}(N) \leq 0\) if and only if \(P\) is decreasing in \((t_0, t_1]\) as \(t\) decreases.

**Proof.** (i) If \(\mathcal{E}(N) \geq 0\), then \(N\) is a lower solution to (1). By Theorem 5, \(P(t) \geq N\) for all \(t \in (t_0, t_1]\). For any number \(\tau \in (0, t_1 - t_0]\), define \(P_\tau : (t_0 + \tau, t_1 + \tau] \to \mathbb{S}^n\) by \(P_\tau(t) = P(t - \tau)\). Since (1) is time-invariant, \(P_\tau(t)\) is a solution to (1) with \(P_\tau(t_1) = P(t_1 - \tau) \geq N = P(t_1)\). By Theorem 5 again, \(P_\tau(t) \geq P(t)\) for \(t \in (t_0 + \tau, t_1]\), or equivalently, \(P(t - \tau) \geq P(t)\) for every \(\tau \in (0, t_1 - t_0]\). In other words, \(P(t)\) is increasing in \((t_0, t_1]\) as \(t\) decreases. Part (ii) is proved similarly by using the fact that \(N\) is an upper solution. \(\Box\)

Theorem 7 implies that if \(P\) is a bounded solution to (1) on \((-\infty, t_1]\) with \(N\) being an upper or lower solution to (2), then \(P_\infty \equiv \lim_{t \to -\infty} P(t)\) exists and \(P_\infty\) is a solution to (2). As a result, we have the following necessary and sufficient existence condition for solutions to (2).

**Theorem 8.** Equation (2) has a solution \(P \in \mathbb{S}^n\) with \(\mathcal{R}(P) > 0\) (\(\mathcal{R}(P) < 0\)) if and only if it has an upper solution \(Y\) and a lower solution \(Z\) such that \(Y \geq Z\) with \(\mathcal{R}(Y) > 0\) (\(\mathcal{R}(Y) < 0\), respectively).

**Proof.** The necessity is obvious by choosing \(Y = Z = P\). For the sufficiency, consider equation (1) with \(N = Y\) and \(Z\) respectively. Since \(Y\) is an upper solution and \(Z\) is a lower solution (1) in \((-\infty, t_1]\), by Theorem 5, there exist solutions \(P_Y\) and \(P_Z\) on \((-\infty, t_1]\) such that \(P_Y(t_1) = Y\), \(P_Z(t_1) = Z\) and \(Y \geq P_Y \geq P_Z \geq Z\). By Theorem 7, both \(P_Y\) and \(P_Z\) are monotone. So \(Y_\infty = \lim_{t \to -\infty} P_Y(t)\) and \(Z_\infty = \lim_{t \to -\infty} P_Z(t)\) exist. Clearly \(Y_\infty\) and \(Z_\infty\) are constant solutions to (2) satisfying \(Y \geq Y_\infty \geq Z_\infty \geq Z\). If \(\mathcal{R}(Z) > 0\) then \(\mathcal{R}(Y_\infty) \geq \mathcal{R}(Z_\infty) > 0\). If \(\mathcal{R}(Y) < 0\) then \(0 \geq \mathcal{R}(Y_\infty) \geq \mathcal{R}(Z_\infty)\). \(\Box\)

In fact, \(Y_\infty\) and \(Z_\infty\) are the maximal and minimal solutions of (2) in the "interval" \([Z, Y] = \{P \in \mathbb{S}^n, Z \leq P \leq Y\}\), respectively. Indeed, if \(M \in [Z, Y]\) is a solution to (2), then \(Y \geq P_Y(t) \geq M \geq P_Z(t) \geq Z\), which imply that \(Y_\infty \geq M \geq Z_\infty\).

We now define stabilizability and detectability for equation (2). Since equation (2) may arise from problems with different state equations, we will avoid using the state equations in our definitions of stabilizability and detectability.

**Definition 2.** Suppose \(A, B, C, D, G, \Pi\) are as in (5) and \(P \in \mathbb{S}^n\). We define the following terms.
(A, C, Π) is ms-stable if there exists $U \in \mathbb{S}^n$, $U > 0$ such that
\[
L(U) + \Pi(U) = A^TU + UA + C^TUC + \Pi(U) < 0. \tag{20}
\]
In other words, $(A, C, \Pi)$ is ms-stable if $L(P) + \Pi(P) = 0$ has a strict upper solution $U > 0$.

$(A, B, C, D, \Pi)$ is ms-stabilizable if there exists $K \in \mathbb{R}^{k \times n}$ such that $(A - BK, C - DK, \Pi)$ is ms-stable, or equivalently, $L(K; P) + \Pi(P)$ has a strict upper solution $U > 0$. Such a matrix $K$ is also called an ms-stabilizing feedback matrix.

$\left( \sqrt{G}, A, C, \Pi \right)$ is ms-detectable if $G = F^TF$ for some $F \in \mathbb{R}^{q \times n}$ and there exist $M_1, M_2 \in \mathbb{R}^{n \times q}$ such that $(A - M_1F, C - M_2F, \Pi)$ is ms-stable.

$P \in \mathbb{S}^n$ is ms-stabilizing if $(A - BK(P), C - DK(P), \Pi)$ is ms-stable.

As proved in [1, Theorem 1], the ms-stabilizability of $(A, B, C, D, 0)$ is equivalent to the existence of a matrix $K \in \mathbb{R}^{k \times n}$ such that the solution $x$ to (12) satisfies $E\{x(t)^2\} \to 0$ as $t \to \infty$; see the paragraph before Proposition 1. When $C = D = 0$, the ms-stability, ms-stabilizability and ms-detectability defined here are equivalent to those defined in [6, Definitions 3.1 and 3.2].

Assuming that $(A, B, C, D, \Pi)$ is ms-stabilizable, we show that (2) has an upper or lower solution and we obtain a refined necessary and sufficient condition for solutions to (2).

**Theorem 9.** Suppose $(A, B, C, D, \Pi)$ is stabilizable.

(i) Equation (2) has a solution $P$ with $\mathcal{R}(P) > 0$ if and only if it has a lower solution $Z$ with $\mathcal{R}(Z) > 0$.

(ii) Equation (2) has a solution $P$ with $\mathcal{R}(P) < 0$ if and only if it has an upper solution $Y$ with $\mathcal{R}(Y) < 0$.

**Proof.** (i) The necessity is trivial. For sufficiency, suppose that $(A, B, C, D, \Pi)$ is ms-stabilizable and that $Z$ is a lower solution with $\mathcal{R}(Z) > 0$. Then $L(K; U) + \Pi(U) < 0$ for some $K \in \mathbb{R}^{k \times n}$ and $U > 0$. Let $Y = \alpha U$. Choose an $\alpha > 0$ such that $Y \geq Z$ and
\[
\mathcal{L}(K; Y) + \Pi(Y) + \mathcal{G}(K) < 0,
\]
where $\mathcal{G}(K)$ is defined in (15). It follows that $\mathcal{R}(Y) \geq \mathcal{R}(Z) > 0$. By Proposition 3 (ii) with $P = Y$, we have that
\[
LQ(Y) + \Pi(Y) \leq \mathcal{L}(K; Y) + \Pi(Y) + \mathcal{G}(K) < 0.
\]
That is, $Y$ is a strict upper solution to (2). By Theorem 8, (2) has a solution in $[Z, Y]$. The proof of (ii) is similar or it follows from Remark 6. □

§ 4. Comparison, Uniqueness, Stabilizability and Approximation of Solutions to (2)

Now we consider the uniqueness, ms-stabilizability and approximation of solutions to (2). We first prove some equivalent descriptions of the ms-stability.

**Theorem 10.**

(i) The following are equivalent.

(a) $(A, C, \Pi)$ is ms-stable.
(b) $\text{Re}(A) < 0$ and $r_\sigma(\tau) < 1$, where $\tau: \mathbb{S}^n \to \mathbb{S}^n$ is defined as $\tau(U) = \int_0^\infty e^{A_t}(C^T U C + \Pi(U)) e^{A_t} dt$.

(c) $L(U) + \Pi(U) + G = 0$ has a unique solution for every $G \in \mathbb{S}^n$. If $G \geq 0$ then $U \geq 0$ and if $G > 0$ then $U > 0$.

(ii) Suppose $(A,C,\Pi)$ is $ms$-stable and $(U,V)$ is a pair of upper and lower solutions to $L(P) + \Pi(P) + G = 0$, then $U \geq V$.

**Proof.** (a) $\Rightarrow$ (b). Suppose $U \in \mathbb{S}^n$, $U > 0$ satisfies $L(U) + \Pi(U) < 0$. Let $H = -[L(U) + \Pi(U)] > 0$, then we have

$$A^T U + U A + C^T U C + \Pi(U) + H = 0.$$  

(21)

In particular, $A^T U + U A < 0$, which implies that $A$ must be stable in the sense that all eigenvalues of $A$ have negative real parts; that is, $\text{Re}(A) < 0$. Now (21) can be rewritten as

$$U = \int_0^\infty e^{A_t}(C^T U C + \Pi(U) + H) e^{A_t} dt = \tau(U) + M,$$

where $M = \int_0^\infty e^{A_t} H e^{A_t} dt > 0$. It follows that for all integers $k \geq 0$, $U = \tau^{k+1}(U) + f_k(M)$, where $f_k(M) \equiv \sum_{i=0}^k \tau^i(M)$. Since $M > 0$, $f_k(M)$ is an increasing sequence and $f_k(M) \leq U$. Therefore, the series $f_\infty(M) \equiv \sum_{i=0}^\infty \tau^i(M)$ converges and $f_\infty(M) = U$. Since each $P \in \mathbb{S}^n$ can be written as $P = P_+ - P_-$ where $P_\pm \geq 0$ and $f_\infty$ is linear, $f_\infty$ is well-defined for all $P \in \mathbb{S}^n$ by $f_\infty(P) = f_\infty(P_+) - f_\infty(P_-)$. Now if $\lambda$ is an eigenvalue of $\tau$, then $\sum_{i=0}^\infty \lambda^i$ must converge to an eigenvalue of $f$. Therefore, $|\lambda| < 1$ and so $r_\sigma(\tau) < 1$.

(b) $\Rightarrow$ (c) Suppose $\text{Re}(A) < 0$ and $r_\sigma(\tau) < 1$. Then $f_\infty(P) = \sum_{i=0}^\infty \tau^i(P)$ is defined for every $P \in \mathbb{S}^n$. Take $P = \int_0^\infty e^{A_t} G e^{A_t} dt$. It is easily checked that $U = f(P)$ satisfies that $U = P + \tau(U)$, which is equivalent to $L(U) + \Pi(U) + G = 0$. Since any solution to $L(U) + \Pi(U) + G = 0$ is represented as $f_\infty(P)$, where $P = \int_0^\infty e^{A_t} G e^{A_t} dt$, the solution has to be unique. If $G \geq 0$ or $> 0$, then $P \geq 0$ or $> 0$, which implies that $U \geq 0$ or $> 0$, respectively.

(c) $\Rightarrow$ (a). Let $G = E$, then $L(U) + \Pi(U) + E = 0$ has a solution $U > 0$. So $(A,C,\Pi)$ is $ms$-stable. This finishes the proof of (i).

To prove (ii), consider $P = U - V$. Then $P$ satisfies $L(P) + \Pi(P) + H = 0$ for some $H \geq 0$. Part (i,c) implies that $U \geq V$.

Now we can prove the following comparison result for (2).

**Theorem 11.** Suppose that $Y$ is an upper solution and $Z$ a lower solution to equation (2). Then $Y \geq Z$ if either (i) $Y$ is $ms$-stabilizing and $\mathcal{R}(Z) > 0$, or (ii) $Z$ is $ms$-stabilizing and $\mathcal{R}(Y) < 0$.

**Proof.** Suppose $Y$ is $ms$-stabilizing and $\mathcal{R}(Z) > 0$. Let $P = Y - Z$ and $H = \mathcal{E}(Z) - \mathcal{E}(Y) \geq 0$. Then $-H = \Pi(P) + \mathcal{L}(Y) - \mathcal{L}(Z)$. By Proposition 3 (i),
\(-H = \Pi(P) + LQ(Y) - LQ(Z) = \\
\Pi(P) + L(K; P) - (K(Y) - K)^T R(Y)(K(Y) - K) + (K(Z) - K)^T R(Z)(K(Z) - K),
\]

where \( L(K; P) = (A - BK)^T P + P(A - BK) + (C - DK)^T P(C - DK) \) as defined in (13).

Setting \( K = K(Y) \) in (22), we get
\[
L(K(Y); P) + \Pi(P) + (K(Z) - K(Y))^T R(Z)(K(Z) - K(Y)) + H = 0.
\]

Since \( R(Z) \geq 0 \) and \( H \geq 0 \), \( P \) is an upper solution to \( L(K(Y); P) + \Pi(P) = 0 \). By assumption, \( (A - BK(Y), C - DK(Y), \Pi) \) is ms-stable. Therefore, we can apply Theorem 10 (ii) to the equation \( L(K(Y); P) + \Pi(P) = 0 \) to conclude that \( P \geq 0 \); that is, \( Y \geq Z \). The proof (ii) is similar or it follows from Remark 6.

Denote by \( Z_+ (Z_-) \) be set of all solutions \( P \in \mathbb{S}^n \) to (2) with \( R(P) > 0 \) ( \( < 0 \), respectively). By Theorem 11, if \( P \in Z_+ \) is ms-stabilizing, then \( P \geq T \) for every \( T \in Z_+ \). It follows that the ms-stabilizing solutions in \( Z_+ \) must be maximal and so unique. Similarly, ms-stabilizing solutions in \( Z_- \) are minimal and unique. Therefore, we have the following uniqueness of ms-stabilizing solutions.

**Proposition 12.**
(i) If \( P \in Z_+ \) is ms-stabilizing, then \( P \) is maximal in \( Z_+ \) and \( P \) is the unique ms-stabilizing solution in \( Z_+ \).
(ii) If \( P \in Z_- \) is ms-stabilizing, then \( P \) is minimal in \( Z_- \) and \( P \) is the unique ms-stabilizing solution in \( Z_- \).

Now we consider the question on the ms-stabilizability of solutions to (2). First consider the linear equation associated with (2):
\[
L(P) + \Pi(P) = A^T P + PA + C^T PC + \Pi(P) + G = 0. \tag{23}
\]

and its generalization with \( K \in \mathbb{R}^{k \times n} \):
\[
L(K; P) + \Pi(P) + G + K^T RK = 0, \tag{24}
\]

where \( L(K; P) \) is defined as in (13). We have

**Theorem 13.** Suppose that \( \left( \sqrt{G}, A, C, \Pi \right) \) is ms-detectable.
(i) If (23) has an upper solution \( P \geq 0 \), then \( (A, C, \Pi) \) is ms-stable.
(ii) More generally, if for some \( K \in \mathbb{R}^{k \times n} \) and \( R \in \mathbb{S}^q \) with \( R > 0 \), (24) has an upper solution \( P \geq 0 \), then \( (A - BK, C - DK, \Pi) \) is ms-stable.

**Proof.** (i) The ms-detectability of \( \left( \sqrt{G}, A, C, \Pi \right) \) implies that \( G = F^T F \) for some \( F \in \mathbb{R}^{q \times n} \) and there exist \( M_1, M_2 \in \mathbb{R}^{n \times q} \) such that \( (A - M_1 F, C - M_2 F, \Pi) \) is ms-stable; that is,
\[
(A - M_1 F)^T V + V(A - M_1 F) + (C - M_2 F)^T V(C - M_2 F) + \Pi(V) < 0 \tag{25}
\]

for some \( 0 < V \in \mathbb{S}^n \). Expanding (25) we obtain
\[
L(V) + \Pi(V) - (F^T M_1^T V + VM_1 F + F^T M_2^T VC + C^T VM_2 F) + F^T M_2^T VM_2 F < 0. \tag{26}
\]
where $L(V) = A^T V + V A + C^T V C$. With $F^T M_2^T V M_2 F \geq 0$ dropped, (26) still holds. It follows that for some $\varepsilon > 0$,

$$L(V) + \Pi(V) - (F^T M_1^T V + V M_1 F + F^T M_2^T V C + C^T V M_2 F) + \varepsilon^2 V + \varepsilon^2 C^T V C < 0.$$  

(27)

Let $a$ be the largest eigenvalue of $(M_1^T V M_1 + M_2^T V M_2)/\varepsilon^2$. Then one has

$$F^T (M_1^T V M_1 + M_2^T V M_2) F/\varepsilon^2 \leq a F^T F \leq -a[L(P) + \Pi(P)],$$

(28)

where the last inequality is just (23). Define $W = a P + V$. Then $W > 0$. Combining (27) and (28), we obtain that

$$L(W) + \Pi(W) = a[L(P) + \Pi(P)] + L(V) + \Pi(V)$$

$$< -[\varepsilon^2 V - F^T M_1^T V - V M_1 F + \varepsilon^2 C^T V C - C^T V M_2 F - F^T M_2^T V C + a F^T F]$$

$$\leq -[(\varepsilon E - M_1 F/\varepsilon)^T V (\varepsilon E - M_1 F/\varepsilon) + (\varepsilon C - M_2 F/\varepsilon)^T V (\varepsilon C - M_2 F/\varepsilon)] \leq 0.$$

This shows that $(A, C, \Pi)$ is ms-stable.

(ii) Let $F = \begin{bmatrix} F & R^{1/2} K \end{bmatrix}$. Then $F^T F = F^T F + K^T R K$. By assumption, $(A - M_1 F, C - M_2 F, \Pi)$ is ms-stable for some $M_1$ and $M_2$. Let $M_1 = [M_1, -BR^{-1/2}]$ and $M_2 = [M_2, -DR^{-1/2}]$. Then we have that

$$A - BK - M_1 F = A - M_1 F, C - DK - M_2 F = C - M_2 F.$$

So $(A - BK - M_1 F, C - DK - M_2 F, \Pi) = (A - M_1 F, C - M_2 F, \Pi)$, which is ms-stable. In other words, $(\sqrt{F^T F}, A - BK, C - DK, \Pi)$ is ms-detectable. By (i) applied to (24), $(A - BK, C - DK, \Pi)$ is ms-stable. The proof (ii) is similar or it follows from Remark 6.□

**Remark 3.** In Theorem 13, a sufficient condition for $(\sqrt{G}, A, C, \Pi)$ to be ms-detectable is that $G > 0$. Indeed, let $F > 0$ such that $G = F^2$ and $\alpha \in (0, \infty)$ such that $\alpha E < -\Pi(E)/2$. Take $M_1 = (A - \alpha E)^{-1}$ and $M_2 = CF^{-1}$. Then the left-hand side of (25) with $V = E$ becomes $2\alpha E + \Pi(E) < 0$. So $(\sqrt{G}, A, C, \Pi)$ is ms-detectable.

Now we prove the ms-stabilizability of solutions to (2).

**Theorem 14.** Suppose $(Y, Z)$ is a pair of upper-lower solutions to (2) and $Y \geq Z$.

(i) If $R(Z) > 0$ and $(\sqrt{F(Z)}, A - BK(Z), C - DK(Z), \Pi)$ is ms-detectable, then $Y$ is ms-stabilizing.

(ii) If $R(Y) < 0$ and $(\sqrt{-F(Y)}, A - BK(Y), C - DK(Y), \Pi)$ is ms-detectable, then $Z$ is ms-stabilizing.

In both cases, (2) has a unique solution $P$ in $[Z, Y]$ and $P$ is ms-stabilizing.

**Proof.** (i) We first assume $S = 0$ and $Z = 0$. The assumptions imply that $G \geq 0$, $R > 0$ and $(\sqrt{G}, A, C, \Pi)$ is ms-detectable. By (16) with $K = K(P)$, equation (2) is equivalent to

$$L(K(P); P) + \Pi(P) + G(K(P)) = 0.$$  

(29)
Since $S = 0$, $G(K(P)) = G + K(P)^T R K(P)$. So (29) is precisely (24) with $K = K(P)$. Because $Y$ is an upper solution to (2) and so it also an upper solution to (29), Theorem 13 (ii) implies that $(A - BK(Y), C - DK(Y), \Pi)$ is ms-stable; that is, $Y$ is ms-stabilizing. For the general case, consider $P = Y - Z$. Then by (19), $P$ satisfies equation (2) is equivalent to

$$\begin{cases}
\hat{G} + \hat{A}^T P + P \hat{A} + \hat{C}^T P \hat{C} + \Pi(P) \\
- (B^T P + D^T P \hat{C})^T (\hat{R} + D^T P D)^+ (B^T P + D^T P \hat{C}) \leq 0, \\
P(t_1) = N - Z(t_1) \geq 0,
\end{cases}$$

(30)

where $\hat{A} = A - BK(Z), \hat{C} = C - DK(Z), \hat{G} = LQ(Z) + \Pi(Z) \geq 0$ and $\hat{R} = R(Z) > 0$. The assumptions imply that (30) has a lower solution 0 with ms-detectable $\left(\sqrt{G}, \hat{A}, \hat{C}, \Pi\right)$. This is precisely the case we just proved with $A, C, G, R$ replaced by $\hat{A}, \hat{C}, \hat{R},$ respectively. Therefore, $P$ is ms-stabilizing in the sense that $\left(\hat{A} - B\hat{K}(P), \hat{C} - D\hat{K}(P), \Pi\right)$ is ms-stable, where $\hat{K}(P) = (\hat{R} + D^T P D)^{-1} (B^T T + D^T P \hat{C})$. Since $\mathcal{R}(Y) \geq \mathcal{R}(Z) > 0$, it is directly checked that $\hat{K}(P) = K(Y) - K(Z)$; see [12, proof of Proposition 5]. It follows that $$(A - BK(Y), C - DK(Y), \Pi) = \left(\hat{A} - B\hat{K}(P), \hat{C} - D\hat{K}(P), \Pi\right)$$ is ms-stable. So $Y$ is ms-stabilizing. In particular, all solutions of (2) in $[Z, Y]$ are ms-stabilizing and so they must be unique by Proposition 12. The proof (ii) is similar or it follows from Remark 6.

By Remark 3, the ms-detectability condition in Theorem 14 hold if $Z$ is a strict lower solution (i.e., $\mathcal{E}(Z) > 0$) or $Y$ is a strict upper solution ($\mathcal{E}(Y) < 0$). Thus we have the following corollary.

**Corollary 15.** Suppose $(Y, Z)$ is a pair of upper-lower solutions and $Y \geq Z$.

(i) If $\mathcal{R}(Z) > 0$ and $\mathcal{E}(Z) > 0$, then $Y$ is ms-stabilizing.

(ii) If $\mathcal{R}(Y) < 0$ and $\mathcal{E}(Y) < 0$, then $Z$ is ms-stabilizing.

In both cases, (2) has a unique solution in $[Z, Y]$ and it is ms-stabilizing.

Finally we show that the stabilizing solution to (2) can be approximated by solutions to linear equations.

**Theorem 16.** Suppose $(Y, Z)$ is a pair of upper and lower solutions such that $Y \geq Z$.

(i) Suppose $\mathcal{R}(Z) > 0$ and $\left(\sqrt{\mathcal{E}(Z)}, A - BK(Z), C - DK(Z), \Pi\right)$ is ms-detectable. Define $H_1 = Y$ and $H_{i+1}$ be the unique solution to

$$\mathcal{L}(\mathcal{K}(H_i); P) + G(\mathcal{K}(H_i)) + \Pi(P) = 0$$

(31)

for $i \geq 1$. Then $H_1 \geq H_2 \geq \cdots \geq Z$ and $H = \lim_{i \to \infty} H_i$ is the unique ms-stabilizing solution to (2) in $[Z, Y]$.

(ii) Suppose $\mathcal{R}(Y) < 0$ and $\left(\sqrt{-\mathcal{E}(Y)}, A - BK(Y), C - DK(Y), \Pi\right)$ is ms-detectable. Define $H_1 = Z$ and $H_{i+1}$ be the solution to (31) for $i \geq 1$. Then $H_1 \leq H_2 \leq \cdots \leq Y$ and $H = \lim_{i \to \infty} H_i$ is the unique ms stabilizing solution to (2) in $[Z, Y]$. 
Proof. (i) We first show that \( H_{i+1} \geq Z \). Since \( Z \) is a lower solution with \( \mathcal{R}(Z) \geq 0 \). By (17.1) with \( P = Z \) and \( K = \mathcal{K}(H_i) \), we have

\[
0 \leq \mathcal{L}(Z) + \Pi(Z) \leq \mathcal{L}(\mathcal{K}(H_i); Z) + \mathcal{G}(\mathcal{K}(H_i)) + \Pi(Z).
\]

It follows that \((H_{i+1}, Z)\) is a pair of upper-lower solution to (31). By Theorem 10 (ii), \( H_{i+1} \geq Z \). Consequently, \( H_{i+1} \) is a solution to (31) with \( \mathcal{R}(H_{i+1}) \geq \mathcal{R}(Z) > 0 \). By (17.1) with \( P = H_{i+1} \) and \( K = \mathcal{K}(H_i) \), we have

\[
\mathcal{L}(H_{i+1}) + \Pi(H_{i+1}) \leq \mathcal{L}(\mathcal{K}(H_i); H_{i+1}) + \mathcal{G}(\mathcal{K}(H_i)) + \Pi(H_{i+1}) = 0.
\]

So \( H_{i+1} \) is an upper solution to (2). By Theorem 14, \( H_{i+1} \) must be ms-stabilizing.

Next we show that \( H_i \geq H_{i+1} \). By (16) with \( P = H_i \), we have

\[
0 \geq \mathcal{L}(H_i) + \Pi(H_i) = \mathcal{L}(\mathcal{K}(H_i); H_i) + \mathcal{G}(\mathcal{K}(H_i)) + \Pi(H_i).
\]

In other words, \((H_i, H_{i+1})\) is a pair of upper-lower solutions to (31). By Theorem 10 (ii) again, \( H_i \geq H_{i+1} \).

Now it is clear that the limit \( H = \lim_{i \to \infty} H_i \) exists and satisfies (2). By Theorem 14, \( H \) is stabilizing. The proof of (ii) is similar or it follows from Remark 6. □

References


