Notes. 1. Each problem is worth 17 points. Choose any 6 problems. Partial/extra credit will be given.
2. For problems 2, 6, 7, you can work on the general problem or the given special case for the full credit.

1. Prove that the symmetric $3 \times 3$ game \[
\begin{array}{ccc}
0 & a & b \\
-a & 0 & c \\
-b & -c & 0
\end{array}
\]
is strictly determined (i.e., has a saddle point) if and only if one of the following three conditions holds. (i) $a \geq 0$ and $b \geq 0$, (ii) $a \geq 0$ and $c \leq 0$, (iii) $b \leq 0$ and $c \leq 0$.

2. Suppose the game in Problem 1 is nonstrictly determined. Show that the maxmin and minmax strategies are

\[
\begin{pmatrix}
\frac{|c|}{|a| + |b| + |c|} & \frac{|b|}{|a| + |b| + |c|} & \frac{|a|}{|a| + |b| + |c|}
\end{pmatrix}.
\]
[If you have difficulty in working on the general case, consider the case \(\{a, b, c\} = \{2, -3, 4\}\).]

3. Prove that a $2 \times 2$ zero-sum game \[
\begin{array}{cc}
a & b \\
c & d
\end{array}
\]
dominates the other, or else one of its rows or columns dominates the other.

4. Prove that if an $m \times n$ game \((a_{ij})\) has two (or more) saddle points, then their payoffs are all equal.

5. Find the value and maxmin and minmax strategies for the game \[
\begin{array}{cc}
-1 & 1 & 2 \\
2 & 1 & -2
\end{array}
\]

6. Let $a, b, c > 0$ and $b^2 > ac$. Show that for each number $r \in \left[\frac{c}{b}, \frac{b}{a}\right]$, \([0, \frac{r}{1+r}, \frac{1}{1+r}, 0]\) is a maxmin strategy of the generalized two-finger Morra game \[
\begin{array}{ccc}
0 & a & -b \\
-a & 0 & b \\
b & 0 & 0 \\
0 & -b & c
\end{array}
\]
[If you have difficulty in working on the general case, consider the case \(\{a, b, c\} = \{2, 3, 4\}\).]

7. Suppose a $2 \times 2$ game \[
\begin{array}{cc}
a & b \\
c & d
\end{array}
\]
has a pair of optimal strategies \(\{1 - x_0, x_0\}\) and \(\{1 - y_0, y_0\}\) that are strictly mixed, that is, \(x_0, y_0 \neq 0\) or 1. Show that for all \(x, y \in [0, 1]\) there holds \(E(x, y_0) = E(x_0, y) = E(x_0, y_0)\) where \(E(x, y)\) is the expected payoff.
[If you have difficulty in working on the general case, consider \[
\begin{array}{cc}
1 & -3 \\
-2 & 2
\end{array}
\]\]