Definitions and Assumptions.

A $m \times n$ **zero-sum** game is an $m \times n$ matrix $(a_{ij})$ of real numbers. It is interpreted as follows.

- There are two players, Ruth and Charlie, in the game. Ruth has $m$ options $\{R_i\}$ and Charlie has $n$ options $\{C_j\}$. The value (payoff or outcome) of the game is $a_{ij}$ if Ruth plays $R_i$ and Charlie plays $C_j$. If $a_{ij} > 0$ then it is a gain (loss) for Ruth (Charlie). If $a_{ij} < 0$ then it is a gain (loss) for Charlie (Ruth).

- A strategy for Ruth is a probability $\{p_i\}$ on $\{R_i\}$, that is, $p_i \geq 0$, $\sum p_i = 1$ and Ruth chooses the option $R_i$ at probability $p_i$. Similarly, a strategy for Charlie is a probability $\{q_j\}$ on $\{C_j\}$, that is, $q_j \geq 0$ and $\sum q_j = 1$.
- Each player is assumed to employ a strategy.

- A strategy $\{p_i\}$ (or $\{q_j\}$) is **pure** if for some $i$ ($j$) $p_i = 1$ ($q_j = 1$) and all others are 0.

- A game is strictly determined if it has a **saddle point** $a_{ij}$, that is, $a_{ij}$ is maximum in the $i$-th row and minimum in the $j$-th column.

- The expected payoff is the average of the values with the "product" probability $\{p_i q_j\}$:

$$E(\{p_i\}, \{q_j\}) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j$$

(1)

- Ruth's goal is to determine a strategy that maximizes the expected payoff:

$$V_l = \max_P \min_Q E(P, Q)$$

while Charlie's goal is to determine a strategy that minimizes the expected payoff:

$$V_u = \min_Q \max_P E(P, Q)$$

It is obvious that

$$V_l \leq V_u.$$  

(2)
Questions:
(1). \( V_l = V_u ? \)
(2). Suppose \( V_l = V_u \), does \( V_l = V_u = E(P_0, Q_0) \) for some pair of strategies \((P_0, Q_0)\)?

**Definition.** If the answers to (1) and (2) are yes, that is,
\[
\max_P \min_Q E(P, Q) = \min_Q \max_P E(P, Q) = E(P_0, Q_0),
\]
then \((P_0, Q_0)\) is called a solution for the game with value \(E(P_0, Q_0)\).

**Theorem I** (von Neumann, 1928). For a general \(m \times n\) zero game with value matrix \((a_{ij})\), there exists a solution \((P_0, Q_0)\) such that (3) holds.

It follows that
\[
E(P, Q) \leq E(P_0, Q_0) \leq E(P_0, Q)
\]

**Lemma.** For each of Ruth's strategy \(\{p_i\}\), Charlie has an optimal pure counter-strategy \(\{q_j\}\) (if Charlie knows Ruth's strategy).

Proof. If \(P = \{p_i\}\) is fixed, then \(E\) is a linear function of \(Q = \{q_j\}\), which has a minimum subject to \(q_j \geq 0\) and \(\sum q_j = 1\). By linear programming, the minimum must occur at the extrema, that is, \(q_j = 1\) for some \(j\) and 0 for other \(j\)'s. In other words, Charlie has a pure optimal pure counter-strategy. \(\Box\)

**Theorem** If \(a_{ij}\) is a saddle point, then
\[
\max_P \min_Q E(P, Q) = \min_Q \max_P E(P, Q) = a_{ij}
\]
which occurs at the pure strategies \(P_0 = (0, \ldots, p_i = 1, \ldots, 0)\) and \(Q_0 = (0, \ldots, q_j = 1, \ldots, 0)\).

Proof. Consider \(i = j = 1\). For any strategies \(P = \{p_i\}\) and \(Q = \{q_j\}\), it follows from the fact that \(a_{ij} = a_{11}\) is a saddle point that
\[
E(P_0, Q_0) = \sum_{i=1}^{m} a_{i1}p_i \geq \sum_{i=1}^{m} a_{11}p_i = a_{11}.
\]
\[
E(P_0, Q) = \sum_{i=1}^{m} a_{1j}q_j \leq \sum_{i=1}^{m} a_{11}q_j = a_{11}.
\]

Note that \(E(P_0, Q_0) = a_{11}\). Therefore, (*) holds. \(\Box\)

**Outline of the Proof of 2 \times 2 case:** If \(\max_P \min_Q E(P, Q) = E(P_0, Q_0)\) and one is pure, then another is also pure. It has to be a saddle point. By the above Theorem 2, there is a pair of pure-strategy solution.
If $P_0$ is mixed, then $Q_0$ is mixed. Then $(P_0, Q_0)$ is the unique critical point.

Some Properties of Matrix Games

1. $\max_P \min_Q E(P, Q) = \min_Q \max_P E(P, Q)$ if and only if there exists $(P_0, Q_0)$ and $v$ such that

$$E(P_0, Q) \geq v \geq E(P_0, Q_0)$$

(this is true in a more general setting regardless the existence of optimal strategies. The proof uses the continuity of $\min_Q E(P, Q)$ in $P$.)

2. It turns out that if $(P_0, Q_0, v)$ satisfies the relationship in 1, then it is a solution to the game and $v = E(P_0, Q_0)$.

3. If there are more than one pair of solutions, then the values are the same.

4. The set of optimal strategies is a convex and bounded set.

5. It turns out that if $(P_0, Q_0, v)$ satisfies the relationship in 1 for pure strategies $P$ and $Q$, then it is a solution to the game and $v = E(P_0, Q_0)$.

6. Suppose $(P_0, Q_0, v)$ is a solution, that is, 1 holds.

   If $v < E(P_0, Q_j)$ then $Q_0$ does not contain the pure strategy $Q_j$.

   If $v > E(P_i, Q_0)$ then $P_0$ does not contain the pure strategy $P_i$.

(Proof. If it did, then a new strategy without the pure strategy will do better.)

7. If a pure strategy is dominated by another strategy (pure or mixed), then it is not contained in the optimal strategies.

8. If a game is symmetric, that is, $A$ is skew-symmetric, then the value is 0 and the optimal strategies are identical: $P_0 = Q_0$. 
Nonzero-Sum Game

\[(a_{ij}, b_{ij})_{m \times n}\]

**Nash Theorem.** There exist strategies \((P_0, Q_0)\) such that for arbitrary strategies \((P, Q)\)

\[E_R(P, Q_0) \leq E_R(P_0, Q_0), \ E_C(P_0, Q) \leq E_C(P_0, Q_0). \quad (*)\]

As before, to verify whether a given pair of strategies \((P_0, Q_0)\) is optimal, we only need to verify (*) against pure strategies.

**Proof of the Nash Theorem** with \(m = n = 2\).

\[E_R(P, Q) = \sum p_ip_ja_{ij}, \ E_C(P, Q) = \sum q_iq_jb_{ij},\]

In the case \(m = n = 2\) we have that for each \(p \in [0, 1]\), \(E_R(p, q)\) is minimum at \(q = 0\) or \(1\). The set

\[\{(p, q) | p \in [0, 1], \text{q is a minimum of } E_R(p, q)\}\]

is a curve from \((0, 0)\) or \((0, 1)\) to \((1, 0)\) or \((1, 1)\). Similarly the set

\[\{(p, q) | q \in [0, 1], \text{p is a minimum of } E_C(p, q)\}\]

we get a curve of points \((p, q)\) from \((0, 0)\) or \((0, 1)\) to \((1, 0)\) or \((1, 1)\). The intersection (which may not be unique) could be at one of the corners, which must the a saddle point. (Again the fact is that if \((p, q)\) is an intersection and one of \(p\) and \(q\) is pure, then another can also be pure.) If not, the it must be a critical point \((p_1, q_1)\) such that

\[\partial_q E_R(p_1, q_1) = \partial_p E_C(p_1, q_1) = 0,\] which implies that \(E_R(p_1, q)\) is independent of \(q\) and \(E_C(p, q_1)\) is independent of \(p\). Therefore each has no gain by playing other strategies when opponent plays the optimal one.