Keys to Homeworks 1, 2.

HW1: Exercises 1,2,6,8*,12,13,14,15 in Chapter 6
HW2: Exercises 16,17,22,30,31,32,34 in Chapter 6

1. Commonsense argument: The function $e$ is bounded from below ($\geq 0$) and convex (intuitively, it is like a parabola open upward). A critical point must be a global minimum.

Analytic argument: Since $e_{mm} = \sum x_j^2 > 0$, by second derivative test, the critical point must be a local minimum (and so the global one since it is the only one) if the Hessian

$$
\begin{pmatrix}
e_{mm} & e_{mb} \\
e_{bm} & e_{bb}
\end{pmatrix}
$$

has positive determinant. The determinant is

$$
e_{mm}e_{bb} - e_{mb}^2 = n\sum_{j=1}^n x_j^2 - \left(\sum_{j=1}^n x_j\right)^2. $$

The easiest way to see that this is positive is to consider inner product of two vectors in $R^n$: $x = (x_1, \cdots, x_n)$ and $y = (1, \cdots, 1)$. The Cauchy-Schwartz inequality implies that $\langle x,y \rangle^2 \leq \langle x,x \rangle \langle y,y \rangle$ with the equality only when $x$ and $y$ are parallel. Since $x_i$’s are supposed to be distinct, $x$ and $y$ can not be parallel. Therefore $\langle x,y \rangle^2 < \langle x,x \rangle \langle y,y \rangle$ or $\left(\sum_{j=1}^n x_j\right)^2 < n\sum_{j=1}^n x_j^2$.

2. The best fit is $y = -1.9801 + 2.26178 x$ with error $e = 0.346387$.

6. The best fit is $y = -0.5909090909090909 - 0.5503246753246753x + 0.45292207792207795x^2$ with error $e = 0.587662$.

8*. Hint. Consider the case $n = m + 1$. Then note that $\Phi = \begin{pmatrix}x_j^i\end{pmatrix}$ for $0 \leq i, j \leq m$ is a square matrix (Vandemonte Matrix) with determinant $\Pi_{0 \leq j < k \leq m} (x_j - x_k) \neq 0$. Therefore $\Phi$ and $\Phi^T$ are invertible.

12. Directly verify that the norms satisfy the three conditions. Note that three norms are in the family $||x||_p = (\sum |x_j|^p)^{1/p}$.

13. For example, $||x|| = k|x_1| + |x_2| + \cdots + |x_n|$ for $k > 0$.

14. Hint for $||v||_1 \leq \sqrt{n}||v||_2$: Use Schwartz inequality with $u = (1, \cdots, 1)$ and $v$.

15, 16, 17, 22, 30: Discussed in class.

31. $c_n = \sqrt{2}(-1)^{n-1}/(n\pi)$.

32. Discussed in class.

34. $\sqrt{2}/(1 + 2n)$. 