Introduction to Calculus of Variations

Notations. Let \( a, b, c, y_1, y_2 \) be real numbers.

\[
\begin{align*}
C[a, b] &= \text{the set of all continuous functions on } [a, b]; \\
C(y_1, y_2)[a, b] &= \text{the set of } y \in C[a, b] \text{ such that } (y(a), y(b)) = (y_1, y_2); \\
C(y_1, y_2, c)[a, b] &= \text{the set of } y \in C(y_1, y_2)[a, b] \text{ such that } \int_a^b y(t)dt = c; \\
C_0[a, b] &= C(0, 0)[a, b] = \text{the set of } y \in C[a, b] \text{ such that } y(a) = y(b) = 0.
\end{align*}
\]

Problem I. Given \((y_1, y_2)\), minimize

\[
\int_a^b |y'(t)|^2 dt
\]

on \(C(y_1, y_2)[a, b]\), that is, find \( y \in C(y_1, y_2)[a, b]\) such that

\[
\int_a^b |y'(t)|^2 dt \leq \int_a^b |z'(t)|^2 dt
\]

for every \( z \in C(y_1, y_2)[a, b]\).

Problem II. Given \((y_1, y_2, c)\), minimize

\[
\int_a^b |y'(t)|^2 dt
\]

on \(C(y_1, y_2, c)[a, b]\).

Problem III. Given \((y_1, y_2)\), minimize

\[
\int_a^b \sqrt{1 + |y'(t)|^2} dt
\]

on \(C(y_1, y_2)[a, b]\).

Problem IV. Given \((y_1, y_2, c)\), minimize

\[
\int_a^b \sqrt{1 + |y'(t)|^2} dt
\]

on \(C(y_1, y_2, c)[a, b]\).

Lemma 1. Suppose \( f \in C[a, b] \). If

\[
\int_a^b f(t)\phi(t) dt = 0,
\]

then...
for every $\phi \in C_0[a,b]$, then $f(t) = 0$ for every $t \in [a,b]$.

**Lemma 2.** Suppose $f \in C[a,b]$. If

$$\int_a^b f(t)\phi(t)dt = 0,$$

for every $\phi \in C_0[a,b]$ with $\int_a^b \phi(t)dt = 0$, then $f(t) = \text{constant}$ for every $t \in [a,b]$.

**Solution to Problem I.**

Suppose $y$ is a minimum of $\int_a^b |y'(t)|^2dt$ in $C_{(y_1,y_2)}[a,b]$. Let $\phi \in C_0[a,b]$ and $p$ be a real number. Define $z(t) = y(t) + p\phi(t)$. Because $\phi(a) = \phi(b) = 0$, $z(a) = y(a) = y_1$ and $z(b) = y(b) = y_2$. In other words, $z \in C_{(y_1,y_2)}[a,b]$. The minimality of $y$ implies that

$$\int_a^b |y'(t)|^2dt \leq \int_a^b |y'(t) + p\phi'(t)|^2dt.$$ 

Denote the right-hand side by $F(p)$. It follows that the function $F(p)$ of $p$ has a minimum at $p = 0$. By Fermat’s theorem in calculus, $F'(0) = 0$, that is,

$$0 = F'(0) = 2\int_a^b y'(t)\phi'(t)dt = -2\int_a^b y''(t)\phi(t)dt. \quad (3)$$

Here we used the fact $\phi(a) = \phi(b) = 0$ and integration by part in the last equality above. Since (3) holds for every $\phi \in C_0[a,b]$, Lemma 1 implies that $y''(t) = 0$. Thus a minimum $y$ satisfies

$$\begin{cases} y''(t) = 0 \\ y(a) = y_1, \ y(b) = y_2. \end{cases}$$

The solution to this boundary value problem is the line passing through $(a,y_1)$ and $(b,y_2)$.

**Remark.** The idea of calculus of variations is to compare the value of the "functional"

$$\int_a^b |y'(t)|^2dt$$

at a minimum $y$ with its value at $z = y + p\phi$. The admissible "test" functions $\phi$ are those such that $y$ and $z$ are in the same class. This amount to $\phi \in C_0[a,b]$ in Problem I. By integration by part, the derivatives of $\phi$ can be converted to $\phi$ itself. Using a fact like Lemma 1, we obtain a differential equation with appropriate boundary conditions.

**Problem II.** With the same idea, we obtain the following condition

$$0 = F'(0) = 2\int_a^b y'(t)\phi'(t)dt = -2\int_a^b y''(t)\phi(t)dt$$

for every test function $\phi$ such that $y$ and $y + p\phi$ are both in $C_{(y_1,y_2,c)}[a,b]$, which means $\phi \in C_0[a,b]$ with $\int_a^b \phi(t)dt = 0$. By Lemma 2, $y''(t) = 2d$ (constant). It follows $y$
satisfies
\[
\begin{aligned}
y'' &= 2d \text{ or } y(t) = dt^2 + e \, t + g \\
y(a) &= y_1, y(b) = y_2 \\
\int_a^b y(t) \, dt &= c.
\end{aligned}
\tag{4}
\]

**Problem III.** With the same idea, we obtain the following condition

\[
0 = \frac{d}{dp} \int_a^b \sqrt{1 + |y'(t) + p\phi'(t)|^2} \, dt \bigg|_{p=0} = \int_a^b \frac{y' \phi'}{\sqrt{1 + |y|^2}} \, dt = -\int_a^b \left( \frac{y'}{\sqrt{1 + |y|^2}} \right)' \phi \, dt
\]

for every test function \( \phi \in C_0[a, b] \). By Lemma 1, \( \left( \frac{y'}{\sqrt{1 + |y|^2}} \right)' = 0 \). It follows that

\[
\frac{y'}{\sqrt{1 + |y|^2}} = d \text{ or } y' = \sqrt{\frac{d^2}{1 - d^2}} = d_1 \text{ or } y = dt + e. \text{ So again the minimum is the line through the two points.}
\]

**Problem IV.** With the same idea, we obtain the following condition

\[
0 = \frac{d}{dp} \int_a^b \sqrt{1 + |y'(t) + p\phi'(t)|^2} \, dt \bigg|_{p=0} = \int_a^b \frac{y' \phi'}{\sqrt{1 + |y|^2}} \, dt = -\int_a^b \left( \frac{y'}{\sqrt{1 + |y|^2}} \right)' \phi \, dt
\]

for every test function \( \phi \in C_0[a, b] \) with \( \int_a^b \phi(t) \, dt = 0 \). By Lemma 2, \( \left( \frac{y'}{\sqrt{1 + |y|^2}} \right)' = d \) (constant). It follows that

\[
\frac{y'}{\sqrt{1 + |y|^2}} = dt + e \text{ or } y' = \pm \frac{dt + e}{\sqrt{1-(dt+e)^2}} \text{ or }
\]

\[
y = \frac{1}{d} \sqrt{1 - (dt+e)^2} + g \ (\pm \text{ can be absorbed by } d), \text{ where } d, e, g \text{ are determined by the conditions}
\]

\[
y(a) = y_1, y(b) = y_2, \int_a^b y(t) \, dt = c.
\tag{5}
\]

**HW10**

1. Find the function \( y \) such that \( y(\pm 1) = 0, \int_{-1}^1 y(t) \, dt = c \) (c is a given number) and the energy \( \int_{-1}^1 |y'(t)|^2 \, dt \) is minimum. Hint: Use (4) and determine \( d, e, g \).

2. Show that the solution to Problem 4 is part of a circle.
3. Find the solution to Problem IV with \([a, b] = [-1, 1], \ y_1 = y_2 = 0 \) and \(c = 1\). For what else \(c\) does the problem have a solution?

General Problems of Calculus of Variations

Suppose \(F(t, y, z)\) and \(G(t, y)\) are functions defined for \((t, y, z)\) in \([a, b] \times R \times R\) and \((y_1, y_2)\) is given.

**Problem V.** Minimize

\[
\int_a^b F(t, y(t), y'(t)) \, dt
\]

for \(y \in C(y_1, y_2, c)[a, b]\).

**Problem VI.** Minimize

\[
\int_a^b F(t, y(t), y'(t)) \, dt \quad \text{subject to} \quad \int_a^b G(t, y(t)) \, dt = c
\]

for \(y \in C(y_1, y_2, c)[a, b]\).

For Problem V, by using the same idea as in Problem I, we get the following differential equation

\[
\frac{d}{dt} F_y(t, y, y') - F_y(t, y, y') = 0, \ y(a) = y_1, \ y(b) = y_2. \tag{6}
\]

For Problem VI, by using the idea of Lagrange multiplier that says that a minimum of \(\int_a^b F(t, y(t), y'(t)) \, dt \) subject to \(\int_a^b G(t, y(t)) \, dt = c\) must be a critical point of \(\int_a^b (F(t, y(t), y'(t)) + \lambda G(t, y(t))) \, dt\) for some number \(\lambda\). Therefore (6) is satisfied with \(F\) replaced by \(F + \lambda G\). So we obtain

\[
\frac{d}{dt} F_y(t, y, y') - F_y(t, y, y') = \lambda G_y(t, y), \ y(a) = y_1, \ y(b) = y_2, \ \int_a^b G(t, y(t)) \, dt = c.
\]

**HW11**

1. Determine the values of \(c_2\) and \(\rho\) such that \(y(t) = c_2 \sin(\rho t)\) satisfies the conditions

\[
y(-\pi) = y(\pi) = 0, \int_{-\pi}^{\pi} |y|^2 \, dt = 1.
\]

Then calculate the energy \(\int_{-\pi}^{\pi} |y'(t)|^2 \, dt\) of the function \(y(t)\).
2. Suppose \( y_1(t) \) and \( y_2(t) \) are two functions on \([a, b]\) such that
\[
y_i(a) = y_i(b) = 0, \quad y_i''(t) = \lambda_i y_i(t)
\]
for \( i = 1, 2 \) with constants \( \lambda_1 \neq \lambda_2 \). Show that \( y_1 \perp y_2 \) in the sense that
\[
\int_a^b y_1(t)y_2(t)dt = 0.
\]

3*. Lemma 2 states that if \( f(t) \) is a continuous function on \([a, b]\) such that
\[
\int_a^b f(t)\phi(t)dt = 0 \quad \text{for every } \phi \in C_0[a, b] \text{ with } \int_a^b \phi(t)dt = 0 \quad (7)
\]
then \( f(t) = \text{constant} \). Give a proof of this lemma by following the steps below.
(This lemma was used in solving Problem II and IV without using Lagrange multiplier method. It is not need if we use Lagrange multiplier method.)

(a) Define \( \hat{f}(s) = f\left(\frac{b-a}{2\pi} s + \frac{b+a}{2}\right) \) and \( \hat{\phi}(s) = \phi\left(\frac{b-a}{2\pi} s + \frac{b+a}{2}\right) \). Show that (7) is equivalent to
\[
\int_{-\pi}^{\pi} \hat{f}(s)\hat{\phi}(s)ds = 0 \quad \text{for every } \hat{\phi} \in C_0[a, b] \text{ with } \int_{-\pi}^{\pi} \hat{\phi}(s)ds = 0.
\]
In other words, we may assume that \( a = -\pi \) and \( b = \pi \).

(b) For \( f \) and \( \phi \) as in (7) with \( a = -\pi \) and \( b = \pi \). Define
\[
c_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\cos(ns)ds, \quad d_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\sin(ns)ds
\]
\[
e_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(t)\cos(ns)ds, \quad g_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(t)\sin(ns)ds
\]
for \( n = 0, 1, 2, \ldots \). Then
\[
f(t) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty} (c_n\cos(nt) + d_n\sin(nt))
\]
\[
\phi(t) = \frac{1}{2}e_0 + \sum_{n=1}^{\infty} (e_n\cos(nt) + g_n\sin(nt))
\]
and
\[
0 = \int_{-\pi}^{\pi} f(t)\phi(t)dt = \frac{1}{4}c_0 e_0 + \sum_{n=1}^{\infty} (c_n e_n + d_n g_n). \quad (8)
\]
(c) Show that the conditions on $\phi$ implies that

$$e_0 = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^n e_n = 0. \quad (9)$$

It follows that (8) holds for all $e_n$ satisfying (9). Then show that $g_n = 0$ and $c_1 = c_2 = \cdots$ for all $n$. Then show that $f$ must be

$$f(t) = \frac{1}{2} e_0 + c_1 \sum_{n=1}^{\infty} \cos(nt).$$

Finally show that $c_1 = 0$ because otherwise the $\int_{-\pi}^{\pi} |f|^2 \, dt$ would be $\infty$, a contradiction to the fact that $f$ is continuous, which implies that $\int_{-\pi}^{\pi} |f|^2 \, dt$ is finite.