Review for Midterm III.

Review all of the Concept Check questions on pp. 637-638, the True-False Quiz questions on p. 638 and try the exercises 1-49,51. The answers to all of these questions will be given during the review session on Wednesday.

Typical problems in Chapter 8.
For numeric series $\sum a_n$:

1. Determine whether the series $\sum a_n$ is convergent or divergent. (Explain how the method is applied.)
   (a). If the series is a geometric series with common ratio $r$ and first term $a$, then the series converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$.
   (b). If the series is a p-series, then it converges if $p > 1$ and it diverges if $p \leq 1$.
   (b). If $a_n$ has the form $(-1)^nb_n$ or $(-1)^{n-1}b_n$ with $b_n > 0$ then the series is an alternating series. It converges if $b_n \downarrow 0$. It diverges if $\lim_{n \to \infty} b_n = l \neq 0$ or $b_n \uparrow$.
   (c). If the series is positive, then the following methods can be used to find out the convergence or divergence.
      (i). Limit comparison test. For examples, if $\lim_{n \to \infty} \frac{a_n}{b_n} = L \neq 0$, then $\sum a_n$ converges if $|r| < 1$ and diverges if $|r| \geq 1$. Similarly if $\lim_{n \to \infty} \frac{a_n}{b_n} = L \neq 0$, then $\sum a_n$ converges if $p > 1$ and diverges if $p \leq 1$.
      (ii). Comparison test. Review the statements and examples in the book.
      (iii). Integral test, which says in the case $f(x) \downarrow 0$ and $a_n = f(n)$ then $\sum a_n$ converges exactly when the integral $\int_1^{\infty} f(x) dx$ converges.
      (iv). Ratio test, which says that if $L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} < 1$ then $\sum a_n$ converges and if $L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1$ then we have to use other method to determine whether the series converges or diverges.
   (d) If the series is not positive, then consider the absolute series $\sum |a_n|$. There are three possibilities.
      (i) $\sum |a_n|$ converges. Then $\sum a_n$ also converges and is said to be convergent absolutely.
      (ii) $\sum |a_n|$ diverges but $\sum a_n$ converges. Then $\sum a_n$ is said to be convergent conditionally, e.g., $\sum (-1)^n \frac{1}{n}$.
      (iii) Both $\sum |a_n|$ and $\sum a_n$ diverges. This happens, for example, $\lim_{n \to \infty} |a_n| \neq 0$ or $\lim_{n \to \infty} |a_{n+1}| > 1$.

2. Estimate the sum of a convergent series and the error of an approximation. When a $\sum_{n=1}^{\infty} a_n$ converges to $s$ and $s_n$ is the partial sum of the first $n$ terms, then the error
   \[ |s - s_n| = |a_{n+1} + \cdots| \]
   can be estimated in the following cases.
For power series \( \sum c_n(x - a)^n \):

3. Find the radius and interval of convergence. If \( L = \lim_{n \to \infty} \frac{|c_{n+1}|}{|c_n|} \). Then the radius of convergence of \( \sum_{n=1}^{\infty} c_n(x - a)^n \) is \( R = 1/L \). The interval of convergence is \((a - R, a + R)\) or \((a - R, a + R)\) or \([a - R, a + R]\).

4. Find the Taylor series of a given function or the Taylor polynomial \( T_n(x) \) of a function of given degree \( n \). There are two methods to do this: by formula and manipulations (substitution, differentiation, integrations). For Taylor polynomial, calculator can be also used.

5. Error estimate for \( |f(x) - T_n(x)| \) for \( |x - a| \leq R \), where \( T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x - a)^i \):

\[
|f(x) - T_n(x)| \leq \frac{M_{n+1}(R)|x-a|^{n+1}}{(n+1)!}
\]

where \( M_{n+1}(R) \) is the maximum of \( |f^{(n+1)}(x)| \) for \( |x - a| \leq R \). This formula can be used in three ways.

(i) to find the error bound for \( |f(x) - T_n(x)| \) when \( n \) and \( R \) are given. We have

\[
|f(x) - T_n(x)| \leq \frac{M_{n+1}(R)R^{n+1}}{(n+1)!}
\]

(ii) determine the degree \( n \) so that

\[
|f(x) - T_n(x)| \leq \frac{M_{n+1}(R)R^{n+1}}{(n+1)!} \leq \epsilon
\]

if \( R \) and the tolerance \( \epsilon \) are given.

(iii) to find the \( R \) such that

\[
|f(x) - T_n(x)| \leq \frac{M_{n+1}(R)R^{n+1}}{(n+1)!} \leq \epsilon
\]

when \( n \) and tolerance \( \epsilon \) are given.