General Algebraic and Differential Riccati Equations from Stochastic LQR Problems with Infinite Horizon

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Abstract. This is a continuation of the paper [12]. We consider general matrix Riccati equations, including those from stochastic linear regulator problems with infinite horizon. For differential Riccati equations, we prove a monotonicity of solutions, which leads to a necessary and sufficient condition for the existence of solutions to algebraic Riccati equations. For solutions to the algebraic Riccati equations, we obtain results on their comparison, uniqueness, stabilizability and approximation.

§ 1. Introduction

The following differential Riccati equation has been studied in [12] by using the method of upper and lower solutions.

$$\begin{cases} P' + A^T P + PA + C^T PC + G + \Pi(P) \\ -(B^T P + D^T PC + S)^T (R + D^T PD)^{-1} (B^T P + D^T PC + S) = 0, \\ P(t_1) = N, \end{cases} \tag{1}$$

where A^T is the transpose of A, $P' = \frac{dP}{dt}$, N is a symmetric matrix, Π is a linear map of symmetric matrices, and A, B, C, D, G, R and S are bounded and measurable matrix functions with appropriate dimensions. The main results in [12] include an interpretation of upper and lower solutions, comparison theorems, an upper-lower solution theorem, necessary and sufficient conditions for existence of solutions, an estimation of maximal existence intervals of solutions and an approximation of solutions.

This paper is a continuation of [12] with focus on the algebraic equation associated with (1):

$$\begin{aligned} &A^T P + PA + C^T PC + G + \Pi(P) \\ &-(B^T P + D^T PC + S)^T (R + D^T PD)^{-1} (B^T P + D^T PC + S) = 0, \end{aligned} \tag{2}$$

where A, B, C, D, G, R and S are constant matrices with appropriate dimensions; see (4) below.

The inclusion of the term Π is important to stochastic control problems with Markovian jumping noises and differential game problems with noises depending on both of the and control; see

[18] and [11], for example. If $\Pi = 0$, then equation (2) becomes

$$\begin{aligned} &A^T P + PA + C^T PC + G \\ &- (B^T P + D^T PC + S)^T (R + D^T PD)^{-1} (B^T P + D^T PC + S) = 0. \end{aligned} \tag{3}$$

This equation arises from a stochastic linear quadratic regulator (LQR) problem of infinite horizon with control-dependent noises; see Problem (11) below in Section 2.

A monotonicity property for solutions to equation (1) will be proved, which leads to necessary and sufficient conditions for the existence of solutions to (2). For solutions to equation (2), we will prove results on comparison, uniqueness, stabilizability and approximation. Several of our results are new while others are generalizations of known results for special cases. See [1] and [23] for some results on equation (2), [6], [7] and [22] for results on equation (1) with $C = D = S = 0$, and [2], [3], [4], [5], [9], [10], [15], [16], [17], [19], [20], [21] and [25] for results on classical Riccati equations.

Our method uses upper and lower solutions to equations (1), (2) and (3), which satisfy the inequalities associated these equations. This method is closely related to the methods of linear matrix inequalities in [1] [19] and semidefinite programming in [23]. As pointed out in [12], our method has several desirable merits. For example, it directly links equation (3) with the LQR problem (11); see Theorem 2 below. It derives our main results under general assumptions. It gives verifiable necessary and sufficient conditions for the existence of solutions (Theorems 8 and 9). It also leads to algorithms for approximating solutions (Theorem 16). In [12], this method is used to estimate the maximal existence intervals of solutions to differential Riccati equations. It applies to both differential and algebraic Riccati equations.

The paper is organized as follows. In Section 2, we introduce some notations and define upper and lower solutions. In addition, we describe the LQR problem (11) and interpret upper and lower solutions to (3) in Theorem 2. Section 2 ends with some results from [12] that are needed in this paper. Specifically, Theorem 5 is an upper-lower solution theorem for equation (1) on a finite interval. Propositions 3 and 4 are some structural properties of equations (1), (2) and (3).

In Section 3, we prove a monotonicity (Theorem 7) for solutions to (1), which leads to a general necessary and sufficient condition (Theorem 8) for the existence of solutions to (2). A simpler existence result (Theorem 9) for (2) is proved under ms-stabilizability. Theorem 9 generalizes a main result in [1, Theorem 10] for (3) with $S = 0$. The definitions of ms-stability, ms-stabilizability and ms-detectability for equation (2) are also given in this section.

In Section 4, we study the comparison, uniqueness, stabilizability and approximation of solutions to (2). Theorem 11 is a comparison theorem for ms-stabilizing solutions. Theorem 12 shows the uniqueness and extreme properties of stabilizing solutions. Generalizing a classical relationship between detectability and stability, Theorem 14 shows that ms-detectability implies the ms-stabilizability of solutions to (2). Theorem 16 gives algorithms for approximating solutions to (2).

§ 2. Preliminary Results

We start with some notations used in this paper. For reader's convenience, most frequently used notations are collected in the Appendix.

Denote $\mathbb{S}^n = \{P \in \mathbb{R}^{n\times n} : P^T = P\}$, where P^T is the transpose of P. We write $M_1 \geq M_2$ ($M_1 > M_2$) if $M_1 - M_2 \in \mathbb{S}^n$ is a positive semidefinite (definite). For a map $\Pi : \mathbb{S}^n \to \mathbb{S}^n$ we write $\Pi \geq 0$ if $\Pi(M) \geq 0$ for each $M \geq 0$.

Assumption. We assume that A, B, C, D, G, R, S and N in equations (1), (2) and (3) are constant matrices and $\Pi : \mathbb{S}^n \to \mathbb{S}^n$ is a linear map, which satisfy

$$A, C \in \mathbb{R}^{n\times n};\ B, D, S^T \in \mathbb{R}^{n\times k}; R \in \mathbb{S}^k;\ G, N \in \mathbb{S}^n;\ \Pi \geq 0. \tag{4}$$

For a Hilbert space $\mathbb{X}$ and an interval I, $L^\infty(I, \mathbb{X})$ is the space of all bounded and measurable functions from I to $\mathbb{X}$. Furthermore, we define $L^{1,\infty}(I, \mathbb{X}) = \{P \in L^\infty(I, \mathbb{X}), P' \in L^\infty(I, \mathbb{X})\}$. The solution P to (1) is assumed to be in $L^{1,\infty}(I, \mathbb{S}^n)$. Since all matrices in (1) are constant, a solution P on an interval is actually smooth. The solution P to (2) and (3) is assumed to be in $\mathbb{S}^n$, which may be considered as a constant solution to the associated differential equation (1).

As in [12], we abbreviate (1), (2) and (3) as

$$\begin{cases} P' + \mathrm{LQ}(P) + \Pi(P) = 0,\ P(t_1) = N, & (1)\\ \mathrm{LQ}(P) + \Pi(P) = 0, & (2)\\ \mathrm{LQ}(P) = 0, & (3)\end{cases}$$

where $\mathrm{LQ}(P) = G + \mathrm{L}(P) - \mathrm{Q}(P)$ and

$$\begin{cases} \mathrm{L}(P) = A^T P + PA + C^T PC,\\ \mathrm{Q}(P) = (B^T P + D^T PC + S)^T (R + D^T PD)^{-1} (B^T P + D^T PC + S)\end{cases} \tag{5}$$

We remark that $\mathrm{Q}(P)$ and $\mathrm{LQ}(P)$ may make sense even if $R + D^T PD$ is singular. To see this, we introduce the following notations

$$\mathcal{R}(P) = R + D^T PD,\ \mathcal{S}(P) = B^T P + D^T PC + S. \tag{6}$$

$$\mathcal{K}(P) = \begin{cases} \mathcal{R}(P)^{-1}\mathcal{S}(P) & \text{if } \mathcal{R}(P) \text{ is nonsingular,}\\ \mathcal{R}(P)^{+}\mathcal{S}(P) & \text{if } \mathcal{R}(P) \text{ is singular,}\end{cases} \tag{7}$$

where $\mathcal{R}(P)^+$ is the pseudoinverse of $\mathcal{R}(P)$. Recall that any matrix M has a unique Moore-Penrose pseudoinverse M^+ with the following properties (see [14] and [1]):

$$\begin{array}{l} MM^+M = M,\ M^+MM^+ = M^+. \\ \text{If } M \in \mathbb{S}^n, \text{ then } M^+ \in \mathbb{S}^n \text{ and } MM^+ = M^+M. \\ M \geq 0 \text{ if and only if } M^+ \geq 0. \end{array} \tag{8}$$

For a function $P \in L^{1,\infty}(I, \mathbb{S}^n)$, $\mathcal{K}(P) = \mathcal{R}(P)^+\mathcal{S}(P)$ always exists, but it may not be bounded on I nor satisfy

$$\mathcal{S}(P) = \mathcal{R}(P)\mathcal{K}(P). \tag{9}$$

We recall the following definitions given in [12]. A function $P \in L^{1,\infty}(I, \mathbb{S}^n)$ is said to be *feasible* if $\mathcal{K}(P) \in L^\infty(I, \mathbb{R}^{k\times n})$ and (9) holds. A function $K \in L^\infty(I, \mathbb{R}^{k\times n})$ is called a *feedback matrix associated with* $P \in L^{1,\infty}(I, \mathbb{S}^n)$ if it satisfies $\mathcal{S}(P) = \mathcal{R}(P)K$. The set of all such $K's$ is denoted by $\mathbb{K}(P)$. Suppose $\mathcal{K}(P) \in L^\infty(I, \mathbb{R}^{k\times n})$, then P is *feasible* if and only if $\mathbb{K}(P) \neq \emptyset$, and in this case,

$$\text{Q}(P) = \mathcal{K}(P)^T\mathcal{R}(P)\mathcal{K}(P) = K^T\mathcal{R}(P)K, \tag{10}$$

for each $K \in \mathbb{K}(P)$; see [12] for proofs of these relationships. In short, $\text{Q}(P)$ and $\text{LQ}(P)$ are well-defined by (10) whenever P is feasible.

If $P \in \mathbb{S}^n$ is constant, then P is *feasible* as long as (9) holds. The definition of "feasible" consists with the term defined in [23] for the associated semidefinite programming problem.

Definition 1. $P \in L^{1,\infty}(I, \mathbb{S}^n)$ is a *solution (upper solution, lower solution)* to (1) if

$$\text{LQ}(P) + \text{LQ}(P) + \Pi(P) = 0\ (\ \leq 0,\ \geq 0),\ P(t_1) = N\ (\ \geq N,\ \leq N).$$

An upper (or lower) solution is *strict* if one of the inequalities is strict. Similarly, $P \in \mathbb{S}^n$ is a *solution (upper, lower solution)* to (2) if $\text{LQ}(P) + \Pi(P) = 0$ (≤ 0, ≥ 0, respectively).

We now briefly describe the LQR problem that leads to equation (3). For a detailed account of this problem, see [1] and [23]. Consider

$$\begin{cases} \text{minimize/maximize } J(u) \text{ for } u \in \mathcal{U}[0,\infty), \text{ where} & \\ J(u) = E\{\int_0^\infty (x^TGx + 2x^TSu + u^TRu)dt\} \text{ subject to} & (11.1) \\ dx = (Ax + Bu)dt + (Cx + Du)dW,\ t \geq 0;\ x(0) = z, & (11.2) \end{cases} \tag{11}$$

where $W(t)$ is a standard Brownian motion for $0 \leq t < \infty$ on a complete probability space with $W(0) = 0$ almost surely, $E\{\}$ is the expectation of the enclosed variable, and $\mathcal{U}[0,\infty)$ is the set of all admissible control processes u defined below. Let $L^2(\mathbb{R}^k)$ (same as $L^2_{\mathcal{F}}(\mathbb{R}^{n_u})$ in [1]) be the space of $\mathbb{R}^k$-valued processes u on $[0,\infty)$ that are adapted to the σ-field generated by $W(t)$ and satisfy $E\int_0^\infty |u|^2dt < \infty$ (square-integrable). Each $u \in L^2(\mathbb{R}^k)$ is an open-loop control. We say that $u \in L^2(\mathbb{R}^k)$ is *ms-stabilizing* if for every $z \in \mathbb{R}^n$ the solution x to equation (11.2) satisfies

$E\{|x(t)|^2\} \to 0$ as $t \to \infty$. The state equation (11.2) is *stabilizable* if there exists a feedback matrix $K \in \mathbb{R}^{k\times n}$ such that $u = -Kx$ is stabilizing, where x is the solution to

$$dx = (A - BK)xdt + (C - DK)xdW,\ x(0) = z, \tag{12}$$

which is induced from (11.2) by $u = -Kx$. In this case, K is called a *stabilizing feedback matrix*. A solution $P \in \mathbb{S}^n$ to (3) is said to be *stabilizing* if $u = -\mathcal{K}(P)x$ is stabilizing.

These concepts will be generalized for equation (2) without referring to the state equation (11.2).

The admissible set $\mathcal{U}[0,\infty)$ consists of all $u \in L^2(\mathbb{R}^k)$ such that the solution x to equation (11.2) is L^2-integrable; that is, $E\{\int_0^\infty |x(t)|^2 dt\} < \infty$. Clearly $J(u)$ is well-defined for each $u \in \mathcal{U}[0,\infty)$. Furthermore, the following relationship holds.

Proposition 1. *Each* $u \in \mathcal{U}[0,\infty)$ *is stabilizing. Conversely, if* $K \in \mathbb{R}^{k\times n}$ such that $u = -Kx$ *is stabilizing, then* $u \in \mathcal{U}[0,\infty)$.

Proof. If $u \in \mathcal{U}[0,\infty)$, then the solution x to (11.2) satisfies $E\{\int_0^\infty |x(t)|^2 dt\} < \infty$, which implies that $E\{|x(t)|^2\} \to 0$ as $t \to \infty$. In other words, u is stabilizing. Conversely, if $u = -Kx$ is stabilizing, then by [1, Theorem 1], there exists $P \in \mathbb{S}^n$, $P > 0$, such that $\mathcal{L}(K;P) < 0$, where

$$\mathcal{L}(K;P) = (A - BK)^T P + P(A - BK) + (C - DK)^T P(C - DK), \tag{13}$$

which is L(P) under the substitution $(A, C) \to (A - BK, C - DK)$. With $u = -Kx$, equation (11.2) reduces to (12). By the Fundamental Theorem of calculus and Ito's lemma, we have

$$\begin{aligned} E\{x^T(t_1)Px(t_1)\} &= z^T Pz + E\int_0^{t_1} \frac{d}{dt} x^T(t)Px(t)dt \\ &= z^T Pz + E\int_0^{t_1} x^T \mathcal{L}(K;P)xdt. \end{aligned} \tag{14}$$

From (14) and the fact that $E\{|x(t_1)|^2\} \to 0$ as $t_1 \to \infty$, one has that $E\int_0^\infty x^T \mathcal{L}(K;P)xdt$ $= -z^T Pz$. This implies that $E\{\int_0^\infty |x(t)|^2 dt\} < \infty$ because $\mathcal{L}(K;P) < 0$. So $u \in \mathcal{U}[0,\infty)$. □

Through approaches of convex optimization and semidefinite programming, interesting relationships between equation (3) and the LQR problem (11) have established in [1] and [23]. Next theorem reveals a close relationship between upper and lower solutions to (3) and the LQR Problem (11).

Theorem 2. *Suppose* $P \in \mathbb{S}^n$ *is feasible and stabilizing. We have*
(i) If LQ$(P) \ge 0$ *and* $\mathcal{R}(P) \ge 0$, *then* $J(u) \ge z^T Pz$ *for each* $u \in \mathcal{U}[0,\infty)$.
(ii) If LQ$(P) \le 0$ *and* $\mathcal{R}(P) \le 0$, *then* $J(u) \le z^T Pz$ *for each* $u \in \mathcal{U}[0,\infty)$.

(iii) If $LQ(P)=0$ and $\mathcal{R}(P)\geq 0$ (or $\mathcal{R}(P)\leq 0$), then z^TPz is the minimum (maximum, respectively) value of $J(u)$ over $\mathcal{U}[0,\infty)$, which occurs when $u=-Kx$, where $K\in\mathbb{K}(P)$ and x satisfies (12).

Proof. Suppose $u\in\mathcal{U}[0,\infty)$ and x is the solution to equation (11.2). By the Fundamental Theorem of calculus and Ito's formula, we have

$$\begin{aligned} E\{x^T(t_1)Px(t_1)\} &= z^TPz+E\int_0^{t_1}\frac{d}{dt}x^T(t)Px(t)dt \qquad (15)\\ &= z^TPz+E\int_0^{t_1}\{x^T\mathrm{L}(P)x+2u^T(B^TP+D^TPC)x+u^TD^TPDu\}dt, \end{aligned}$$

where $\mathrm{L}(P)=A^TP+PA+C^TPC$ as defined in (5). Since $E\{|x(t_1)|^2\}\to 0$ as $t_1\to\infty$, the limit of (15) becomes

$$0=z^TPz+E\int_0^{\infty}\{x^T\mathrm{L}(P)x+2u^T(B^TP+D^TPC)x+u^TD^TPDu\}dt. \qquad (16)$$

Adding (16) to $J(u)$ and using the notations $\mathcal{R}(P)$ and $\mathcal{S}(P)$ in (6), we obtain

$$J(u)=z^TPz+E\int_0^{\infty}\{x^T(\mathrm{L}(P)+G)x+2u^T\mathcal{S}(P)x+u^T\mathcal{R}(P)u\}dt.$$

Since $\mathcal{S}(P)=\mathcal{R}(P)K$ for each $K\in\mathbb{K}(P)$, we can write

$$2u^T\mathcal{S}(P)x+u^T\mathcal{R}(P)u=(u+Kx)^T\mathcal{R}(P)(u+Kx)-K^T\mathcal{R}(P)K.$$

Recall that $\mathrm{LQ}(P)=G+\mathrm{L}(P)-K^T\mathcal{R}(P)K$. It follows that

$$J(u)=z^TPz+E\int_0^{\infty}\left\{x^T\mathrm{LQ}(P)x+(u+Kx)^T\mathcal{R}(P)(u+Kx)\right\}dt. \qquad (17)$$

In case (i), we have $\mathrm{LQ}(P)\geq 0$ and $\mathcal{R}(P)\geq 0$, so (17) implies that $J(u)\geq z^TPz$ for every $u\in\mathcal{U}[0,\infty)$. Similarly, in case (ii), (17) implies that $J(u)\leq z^TPz$ for every $u\in\mathcal{U}[0,\infty)$. In case (iii), (17) implies that for every $u\in\mathcal{U}[0,\infty)$,

$$J(u)=z^TPz+E\int_0^{\infty}\left\{(u+Kx)^T\mathcal{R}(P)(u+Kx)\right\}dt.$$

It follows that $J(u)$ has a minimum (maximum) z^TPz at $u=-Kx$ if $\mathcal{R}(P)\geq 0$ (if $\mathcal{R}(P)\leq 0$). Equation (12) is precisely the state equation (11.2) with $u=-Kx$. □

Note that when $u=-Kx$, the cost $J(u)$ becomes $J_K=\int_0^{\infty}x^T\mathcal{G}(K)xdt$, where

$$\mathcal{G}(K)=K^TRK-K^TS-S^TK+G. \qquad (18)$$

The following two propositions and Theorem 5 are proved in [12] and will be used in this paper.

Proposition 3. *Suppose* $P \in L^{1,\infty}(I,\mathbb{S}^n)$ *is feasible and* $K \in L^{\infty}(I,\mathbb{R}^{k\times n})$. *Let* $\mathrm{LQ}(P)$, $\mathcal{L}(K;P)$ *and* $\mathcal{G}(K)$ *be defined in (5), (13) and (18), respectively. Then*

(i)
$$\mathrm{LQ}(P) + (\mathcal{K}(P)-K)^T\mathcal{R}(P)(\mathcal{K}(P)-K) = \mathcal{G}(K) + \mathcal{L}(K;P), \tag{19}$$

(ii)
$$\begin{cases} \mathrm{LQ}(P) \le \mathcal{G}(K) + \mathcal{L}(K;P), \text{ if } \mathcal{R}(P) \ge 0 & (20.1) \\ \mathrm{LQ}(P) \ge \mathcal{G}(K) + \mathcal{L}(K;P), \text{ if } \mathcal{R}(P) \le 0 & (20.2) \\ \mathrm{LQ}(P) = \mathcal{G}(\mathcal{K}(P)) + \mathcal{L}(\mathcal{K}(P);P). & (20.3) \end{cases} \tag{20}$$

Proposition 4. *Suppose* $Y, Z \in L^{1,\infty}(I,\mathbb{S}^n)$ *are feasible and* $K \in L^{\infty}(I,\mathbb{R}^{k\times n})$. *Denote* $P = Y - Z$, $\widehat{A} = A - B\mathcal{K}(Z)$, $\widehat{C} = C - D\mathcal{K}(Z)$ *and* $\widehat{R} = \mathcal{R}(Z)$. *Then*

(i)
$$\begin{aligned} &\mathrm{LQ}(Y) - \mathrm{LQ}(Z) \\ &= \widehat{A}^T P + P\widehat{A} + \widehat{C}^T P\widehat{C} - \left(B^T P + D^T P\widehat{C}\right)^T \left(\widehat{R} + D^T P D\right)^+ \left(B^T P + D^T P\widehat{C}\right) \end{aligned} \tag{21}$$

(ii) If Z *is given, then equation (1) for* Y *is equivalent to the following equation for* $P = Y - Z$:

$$\begin{cases} P' + \widehat{G} + \widehat{A}^T P + P\widehat{A} + \widehat{C}^T P\widehat{C} + \Pi(P) \\ \quad - \left(B^T P + D^T P\widehat{C}\right)^T \left(\widehat{R} + D^T P D\right)^+ \left(B^T P + D^T P\widehat{C}\right) = 0, \\ P(t_1) = N - Z(t_1), \end{cases} \tag{22}$$

where $\widehat{G} = Z' + LQ(Z) + \Pi(Z)$.

Note that Z is a lower solution to (1) if and only if $\widehat{G} \ge 0$ and $N - Z(t_1) \ge 0$, that is, 0 is a lower solution to (22). Similarly, Z is an upper solution to (1) if and only if 0 an upper solution to (22). In other words, equation (1) has an upper or lower solutions is equivalent to that (1) can be translated to a *standard* problem that has 0 as an upper or lower solution.

Also note that Propositions 3 and 4 hold, in particular, for P, Y, $Z \in \mathbb{S}^n$ and $K \in \mathbb{R}^{k\times n}$.

Theorem 5 (Upper-lower solution theorem). *Suppose that* (Y, Z) *is a pair of upper-lower solutions to (1) on a finite interval* I.

(i) If either $\mathcal{R}(Z) \ge 0$ *or* $\mathcal{R}(Y) \le 0$, *then* $Y \ge Z$. *In addition, if one of* Y *and* Z *is strict, then* $Y > Z$.

(ii) If either $\mathcal{R}(Z) > 0$ *or* $\mathcal{R}(Y) < 0$, *then equation (1) has a unique solution* P *with* $Y \ge P \ge Z$.

Remark 6. As noted in [12, Remark 3], if a lower solution Z (if it exists) to (1) with $\mathcal{R}(Z) > 0$ has certain property, then an upper solution Y (if it exists) to (1) with $\mathcal{R}(Y) < 0$ also has the corresponding property. Same is true for equations (2) and (3). Because of this, properties of upper solutions may be stated without proof.

§3. Monotonicity of Solutions to (1) and Existence of Solutions to (2)

Now we consider equations (1) and (2), which are

$$\mathcal{E}(P) \equiv P' + \mathrm{LQ}(P) + \Pi(P) = 0, P(t_1) = N, \tag{1}$$

$$\mathcal{E}(P) \equiv \mathrm{LQ}(P) + \Pi(P) = 0. \tag{2}$$

Since a solution $P \in \mathbb{S}^n$ to (2) is also a solution to (1), we use the same notation $\mathcal{E}(P)$ for both equations. By the local theory of differential equations, equation (1) has a unique solution in a maximum interval $(t_0, t_1]$.

Next theorem shows that a solution to (1) is monotone if and only if N is a lower or upper solution to (2); that is, $\mathcal{E}(N) \geq 0$ or $\mathcal{E}(N) \leq 0$.

Theorem 7. *Suppose P is the feasible solution of (1) in $(t_0, t_1]$. Then we have*
(i) $\mathcal{E}(N) \geq 0$ if and only if P is increasing in $(t_0, t_1]$ as t decreases.
(ii) $\mathcal{E}(N) \leq 0$ if and only if P is decreasing in $(t_0, t_1]$ as t decreases.

Proof. (i) If $\mathcal{E}(N) \geq 0$, then N is a lower solution to (1). By Theorem 5, $P(t) \geq N$ for all $t \in (t_0, t_1]$. For any number $\tau \in (0, t_1 - t_0)$, define $P_* : (t_0 + \tau, t_1 + \tau] \to \mathbb{S}^n$ by $P_*(t) = P(t - \tau)$. Since (1) is time-invariant, $P_*(t)$ is a solution to (1) with $P_*(t_1) = P(t_1 - \tau) \geq N = P(t_1)$. By Theorem 5 again, $P_*(t) \geq P(t)$ for $t \in (t_0 + \tau, t_1]$, or equivalently, $P(t - \tau) \geq P(t)$ for every $\tau \in (0, t_1 - t_0)$. In other words, $P(t)$ is increasing in $(t_0, t_1]$ as t decreases. Proof of (ii) is similar using the fact that N is an upper solution.□

Theorem 7 implies that if P is a bounded solution to (1) on $(-\infty, t_1]$ with N being an upper or lower solution to (2), then $P_\infty \equiv \lim_{t \to -\infty} P(t)$ exists and P_∞ is a solution to (2). As a result, we have the following necessary and sufficient existence condition for solutions to (2).

Theorem 8. *Equation (2) has a solution $P \in \mathbb{S}^n$ with $\mathcal{R}(P) > 0$ $(\mathcal{R}(P) < 0)$ if and only if it has a pair (Y, Z) of upper and lower solutions with $Y \geq Z$ and $\mathcal{R}(Z) > 0$ $(\mathcal{R}(Y) < 0$, respectively).*

Proof. The necessity is obvious by choosing $Y = Z = P$. For the sufficiency, consider equation (1) with boundary values $N = Y$ and Z, respectively. Since Y is an upper solution and Z is a lower solution (1) in $(-\infty, t_1]$, by Theorem 5, there exist solutions P_Y and P_Z on $(-\infty, t_1]$ such that $P_Y(t_1) = Y$, $P_Z(t_1) = Z$ and $Y \geq P_Y \geq P_Z \geq Z$. By Theorem 7, both P_Y and P_Z are monotone. So $Y_\infty = \lim_{t\to-\infty} P_Y(t)$ and $Z_\infty = \lim_{t\to-\infty} P_Z(t)$ exist. Clearly Y_∞ and Z_∞ are solutions to (2) and they satisfy $Y \geq Y_\infty \geq Z_\infty \geq Z$. If $\mathcal{R}(Z) > 0$ then $\mathcal{R}(Y_\infty) \geq \mathcal{R}(Z_\infty) > 0$. If $\mathcal{R}(Y) < 0$ then $0 > \mathcal{R}(Y_\infty) \geq \mathcal{R}(Z_\infty)$. □

In fact, Y_∞ and Z_∞ are the maximal and minimal solutions of (2) in the "interval" $[Z, Y] = \{P \in \mathbb{S}^n, Z \leq P \leq Y\}$, respectively. Indeed, if $M \in [Z, Y]$ is a solution to (2), then $Y \geq P_Y(t) \geq M \geq P_Z(t) \geq Z$, which imply that $Y_\infty \geq M \geq Z_\infty$.

Since equation (2) may arise from problems with different state equations, it is more appropriate to introduce the concepts of stabilizability and detectability for equation (2) without referring these state equations. Suppose A, B, C, D, G, Π are as in (4) and $P \in \mathbb{S}^n$. We have

Definition 2. (A, C, Π) is *ms-stable* if there exists $U \in \mathbb{S}^n$, $U > 0$ such that

$$\mathrm{L}(U) + \Pi(U) = A^T U + UA + C^T UC + \Pi(U) < 0; \tag{23}$$

that is, $\mathrm{L}(P) + \Pi(P) = 0$ has a strict upper solution $U > 0$.

(A, B, C, D, Π) is *ms-stabilizable* if there exists $K \in \mathbb{R}^{k\times n}$ such that $(A - BK, C - DK, \Pi)$ is ms-stable; that is, $\mathcal{L}(K; P) + \Pi(P)$ has a strict upper solution $U > 0$. Such a matrix K is called an ms-stabilizing feedback matrix.

$\left(\sqrt{G}, A, C, \Pi\right)$ is *ms-detectable* if $G = F^T F$ for some $F \in \mathbb{R}^{q\times n}$ and there exist $M_1, M_2 \in R^{n\times q}$ such that $(A - M_1 F, C - M_2 F, \Pi)$ is ms-stable.

$P \in \mathbb{S}^n$ is *ms-stabilizing* if $(A - B\mathcal{K}(P), C - D\mathcal{K}(P), \Pi)$ is ms-stable.

As proved in [1, Theorem 1], the *ms-stabilizability of* $(A, B, C, D, 0)$ is equivalent to the existence of a matrix $K \in \mathbb{R}^{k\times n}$ such that the solution x to (12) satisfies $E\{|x(t)|^2\} \to 0$ as $t \to \infty$. When $C = D = 0$, the ms-stabilility, ms-stabilizability and ms-detectability defined here are equivalent to those defined in [6, Definitions 3.1 and 3.2]. So our definitions generalize these classical concepts to equation (2).

Assuming the *ms*-stabilizbility of (A, B, C, D, Π), we obtain a simpler necessary and sufficient condition for solutions to (2) in the next theorem.

Theorem 9. *Suppose* (A, B, C, D, Π) *is stabilizable.*

(i) Equation (2) has a solution P *with* $\mathcal{R}(P) > 0$ *if and only if it has a lower solution* Z *with* $\mathcal{R}(Z) > 0$.

(ii) Equation (2) has a solution P with $\mathcal{R}(P) < 0$ if and only if it has an upper solution Y with $\mathcal{R}(Y) < 0$.

Proof. (i) The necessity is trivial. For sufficiency, suppose that (A, B, C, D, Π) is ms-stabilizable and that Z is a lower solution with $\mathcal{R}(Z) > 0$. Then $\mathcal{L}(K; U) + \Pi(U) < 0$ for some $K \in \mathbb{R}^{k \times n}$ and $U > 0$. Let $Y = \alpha U$. Choose an $\alpha > 0$ such that $Y \geq Z$ and

$$\mathcal{L}(K; Y) + \Pi(Y) + \mathcal{G}(K) = \alpha(\mathcal{L}(K; U) + \Pi(U)) + \mathcal{G}(K) < 0,$$

where $\mathcal{G}(K)$ is defined in (18). It follows that $\mathcal{R}(Y) \geq \mathcal{R}(Z) > 0$. By Proposition 3 (ii) with $P = Y$, we have that

$$\mathrm{LQ}(Y) + \Pi(Y) \leq \mathcal{L}(K; Y) + \Pi(Y) + \mathcal{G}(K) < 0.$$

This shows that Y is a strict upper solution to (2). By Theorem 8, equation (2) has a solution in $[Z, Y]$. The proof of (ii) is similar or it follows from Remark 6. ☐

In the case $C = D = S = 0$, some of the results in this section are well-known; for example, see [6], [7] and [22].

§ 4. Comparison, Uniqueness, Stabilizability and Approximation of Solutions to (2)

Now we consider the uniqueness, ms-stabilizability and approximation of solutions to (2). We first prove some equivalent descriptions of the ms-stability of (A, C, Π). Define τ: $\mathbb{S}^n \to \mathbb{S}^n$ by $\tau(U) = \int_0^\infty e^{A^T t}(C^T U C + \Pi(U)) e^{At} dt$. Denote by $\lambda(A)$ the set of all eigenvalues of A and by $r_\sigma(\tau)$ the spectral radius of τ.

Theorem 10. (i) *The following are equivalent.*

(a) (A, C, Π) *is ms-stable.*

(b) $\mathrm{Re}\lambda(A) < 0$ *and* $r_\sigma(\tau) < 1$.

(c) $\mathrm{L}(U) + \Pi(U) + G = 0$ *has a unique solution for every* $G \in \mathbb{S}^n$*. If* $G \geq 0$ *then* $U \geq 0$ *and if* $G > 0$ *then* $U > 0$.

(ii) *Suppose* (A, C, Π) *is ms-stable and* (U, V) *is a pair of upper and lower solutions to* $L(P) + \Pi(P) + G = 0$*, then* $U \geq V$.

Proof. (a) ⇒ (b). Suppose $U \in \mathbb{S}^n$, $U > 0$ satisfies $\mathrm{L}(U) + \Pi(U) < 0$. Let $H = -[\mathrm{L}(U) + \Pi(U)] > 0$, then we have

$$A^T U + UA + C^T UC + \Pi(U) + H = 0. \tag{24}$$

In particular, $A^T U + UA < 0$, which implies that A must be stable in the sense that all eigenvalues of A have negative real parts; that is, Re$\lambda(A) < 0$. Now (24) can be rewritten as

$$U = \int_0^\infty e^{A^T t}(C^T UC + \Pi(U) + H)e^{At}dt = \tau(U) + M,$$

where $M = \int_0^\infty e^{A^T t} H e^{At} dt > 0$. It follows that for all integers $k \geq 0$, $U = \tau^{k+1}(U) + f_k(M)$, where $f_k(M) \equiv \sum_{i=0}^{k} \tau^i(M)$. Since $M > 0$, $f_k(M)$ is an increasing sequence and $f_k(M) \leq U$. Therefore, the series $f_\infty(M) \equiv \sum_{i=0}^{\infty} \tau^i(M)$ converges and $f_\infty(M) = U$. Since each $P \in \mathbb{S}^n$ can be written as $P = P_+ - P_-$ where $P_\pm \geq 0$ and f_∞ is linear, f_∞ is well-defined for all $P \in \mathbb{S}^n$ by $f_\infty(P) = f_\infty(P_+) - f_\infty(P_-)$. Now if λ is an eigenvalue of τ, then $\sum_{i=0}^{\infty} \lambda^i$ must converge to an eigenvalue of f. Therefore, $|\lambda| < 1$ and so $r_\sigma(\tau) < 1$.

(b) ⇒ (c) Suppose $Re\lambda(A) < 0$ and $r_\sigma(\tau) < 1$. Then $f_\infty(P) = \sum_{i=0}^{\infty} \tau^i(P)$ is defined for every $P \in \mathbb{S}^n$. Take $P = \int_0^\infty e^{A^T t} G e^{At} dt$. It is easily checked that $U = f(P)$ satisfies that $U = P + \tau(U)$, which is equivalent to $\mathrm{L}(U) + \Pi(U) + G = 0$. Since any solution to $\mathrm{L}(U) + \Pi(U) + G = 0$ is represented as $f_\infty(P)$, where $P = \int_0^\infty e^{A^T t} G e^{At} dt$, the solution has to be unique. If $G \geq 0$ or > 0, then $P \geq 0$ or > 0, which implies that $U \geq 0$ or > 0, respectively.

(c) ⇒ (a). Let $G = E$, then $\mathrm{L}(U) + \Pi(U) + E = 0$ has a solution $U > 0$. So (A, C, Π) is ms-stable. This finishes the proof of (i).

To prove (ii), consider $P = U - V$. Then P satisfies $\mathrm{L}(P) + \Pi(P) + H = 0$ for some $H \geq 0$. Part (i.c) implies that $U \geq V$.□

The idea of the proof of Theorem 10(i) is from [22] and [6], which proved (i) for the case $C = D = S = 0$.

Unlike (1), the solutions of (2) may not be unique. Nevertheless, we have the following comparison result for (2).

Theorem 11. *Suppose that Y is an upper solution and Z a lower solution to equation (2). Then $Y \geq Z$ if either (i) Y is ms-stabilizing and $\mathcal{R}(Z) > 0$, or (ii) Z is ms-stabilizing and $\mathcal{R}(Y) < 0$.*

Proof. Suppose Y *is ms-stabilizing and* $\mathcal{R}(Z) > 0$. Let $P = Y - Z$ and $H = \mathcal{E}(Z) - \mathcal{E}(Y) \geq 0$. Then $-H = \Pi(P) + \mathrm{LQ}(Y) - \mathrm{LQ}(Z)$. By Proposition 3 (i) with $P = Y$ and $P = Z$, respectively,

$$-H = \Pi(P) + \mathrm{LQ}(Y) - \mathrm{LQ}(Z) = \tag{25}$$
$$\Pi(P) + \mathcal{L}(K;P) - (\mathcal{K}(Y) - K)^T \mathcal{R}(Y)(\mathcal{K}(Y) - K) + (\mathcal{K}(Z) - K)^T \mathcal{R}(Z)(\mathcal{K}(Z) - K),$$

where $\mathcal{L}(K;P) = (A - BK)^T P + P(A - BK) + (C - DK)^T P(C - DK)$ as defined in (13). Setting $K = \mathcal{K}(Y)$ in (25), we get

$$\mathcal{L}(\mathcal{K}(Y);P) + \Pi(P) + (\mathcal{K}(Z) - \mathcal{K}(Y))^T \mathcal{R}(Z)(\mathcal{K}(Z) - \mathcal{K}(Y)) + H = 0.$$

Since $\mathcal{R}(Z) \geq 0$ and $H \geq 0$, P is an upper solution to $\mathcal{L}(\mathcal{K}(Y);P) + \Pi(P) = 0$. By the assumption that Y is ms-stabilizing, $(A - B\mathcal{K}(Y), C - D\mathcal{K}(Y), \Pi)$ is ms-stable. Therefore, we can apply Theorem 10 (ii) to the linear equation $\mathcal{L}(\mathcal{K}(Y);P) + \Pi(P) = 0$ to conclude that $P \geq 0$; that is, $Y \geq Z$. The proof (ii) is similar or it follows from Remark 6.☐

Denote by $\mathcal{Z}_+$ ($\mathcal{Z}_-$) be set of all solutions $P \in \mathbb{S}^n$ to (2) with $\mathcal{R}(P) > 0$ (< 0, respectively). By Theorem 11, if $P \in \mathcal{Z}_+$ is ms-stabilizing, then $P \geq T$ for every $T \in \mathcal{Z}_+$. It follows that the ms-stabilizing solutions in $\mathcal{Z}_+$ must be maximal and so unique. Similarly, ms-stabilizing solutions in $\mathcal{Z}_-$ are minimal and unique. Therefore, we have the following uniqueness of ms-stabilizing solutions. For the same result for classical Riccati equations ($C = D = S = \Pi = 0$), see [25, Ch. 13] for example.

Proposition 12.
(i) If $P \in \mathcal{Z}_+$ is ms-stabilizing, then P is maximal in $\mathcal{Z}_+$ and P is the unique ms-stabilizing solution in $\mathcal{Z}_+$.
(ii) If $P \in \mathcal{Z}_-$ is ms-stabilizing, then P is minimal in $\mathcal{Z}_-$ and P is the unique ms-stabilizing solution in $\mathcal{Z}_-$.

Now we consider the ms-stabilizability of solutions to (2). First consider the linear equation associated with (2):

$$\mathrm{L}(P) + \Pi(P) \equiv A^T P + PA + C^T PC + \Pi(P) + G = 0. \tag{26}$$

and its generalization with $K \in \mathbb{R}^{k \times n}$:

$$\mathcal{L}(K;P) + \Pi(P) + G + K^T RK = 0, \tag{27}$$

where $\mathcal{L}(K;P)$ is defined as in (13). We have

Theorem 13. *Suppose that $\left(\sqrt{G}, A, C, \Pi\right)$ is ms-detectable.*
(i) *If (26) has an upper solution $P \geq 0$, then (A, C, Π) is ms-stable.*
(ii) *More generally, if for some $K \in R^{k \times n}$ and $R \in \mathbb{S}^n$ with $R > 0$, (27) has an upper solution $P \geq 0$, then $(A - BK, C - DK, \Pi)$ is ms-stable.*

Proof. (i) The ms-detectability of $\left(\sqrt{G}, A, C, \Pi\right)$ implies that $G = F^T F$ for some $F \in \mathbb{R}^{q\times n}$ and there exist $M_1, M_2 \in R^{n\times q}$ such that $(A - M_1 F, C - M_2 F, \Pi)$ is ms-stable; that is,

$$(A - M_1 F)^T V + V(A - M_1 F) + (C - M_2 F)^T V(C - M_2 F) + \Pi(V) < 0 \tag{28}$$

for some $0 < V \in \mathbb{S}^n$. Expanding (28) we obtain

$$\mathrm{L}(V) + \Pi(V) - (F^T M_1^T V + V M_1 F + F^T M_2^T V C + C^T V M_2 F) + F^T M_2^T V M_2 F < 0. \tag{29}$$

where $\mathrm{L}(V) = A^T V + VA + C^T V C$. With $F^T M_2^T V M_2 F \geq 0$ dropped, (29) still holds. It follows that for some $\varepsilon > 0$,

$$\mathrm{L}(V) + \Pi(V) - (F^T M_1^T V + V M_1 F + F^T M_2^T V C + C^T V M_2 F) + \epsilon^2 V + \epsilon^2 C^T V C < 0. \tag{30}$$

Let a be the largest eigenvalue of $(M_1^T V M_1 + M_2^T V M_2)/\varepsilon^2$. Then one has

$$F^T(M_1^T V M_1 + M_2^T V M_2)F/\varepsilon^2 \leq aF^T F = -a[\mathrm{L}(P) + \Pi(P)], \tag{31}$$

where the last inequality is just (26). Define $W = aP + V$. Then $W > 0$. Combining (30) and (31), we obtain that

$$\begin{aligned}
&\mathrm{L}(W) + \Pi(W) = a(\mathrm{L}(P) + \Pi(P)) + \mathrm{L}(V) + \Pi(V) \\
&< -\left[\varepsilon^2 V - F^T M_1^T V - V M_1 F + \varepsilon^2 C^T V C - C^T V M_2 F - F^T M_2^T V C + aF^T F\right] \\
&\leq -\left[(\varepsilon E - M_1 F/\varepsilon)^T V(\varepsilon E - M_1 F/\varepsilon) + (\varepsilon C - M_2 F/\varepsilon)^T V(\varepsilon C - M_2 F/\varepsilon)\right] \leq 0.
\end{aligned}$$

This shows that (A, C, Π) is ms-stable.

(ii) Let $\mathcal{F} = \begin{bmatrix} F \\ R^{1/2}K \end{bmatrix}$. Then $\mathcal{F}^T\mathcal{F} = F^T F + K^T R K$. By assumption, $(A - M_1 F, C - M_2 F, \Pi)$ is ms-stable for some M_1 and M_2. Let $\mathcal{M}_1 = \left[M_1, -BR^{-1/2}\right]$ and $\mathcal{M}_2 = \left[M_2, -DR^{-1/2}\right]$. Then we have that

$$A - BK - \mathcal{M}_1\mathcal{F} = A - M_1 F,\ C - DK - \mathcal{M}_2\mathcal{F} = C - M_2 F.$$

So $(A - BK - \mathcal{M}_1\mathcal{F}, C - DK - \mathcal{M}_2\mathcal{F}, \Pi) = (A - M_1 F, C - M_2 F, \Pi)$, which is ms-stable. In other words, $\left(\sqrt{\mathcal{F}^T\mathcal{F}}, A - BK, C - DK, \Pi\right)$ is ms-detectable. By (i) applied to (27), $(A - BK, C - DK, \Pi)$ is ms-stable. The proof (ii) is similar or it follows from Remark 6.☐

Remark 3. The special case $C = D = S = Z = 0$ of Theorem 13 is proved in [6, Lemmas 3.2, 3.4]. Note that in Theorem 13, $\left(\sqrt{G}, A, C, \Pi\right)$ is ms-detectable if $G > 0$. Indeed, let $F > 0$ such that $G = F^2$ and $\alpha \in (0, \infty)$ such that $\alpha E < -\Pi(E)/2$. Take $M_1 = (A - \alpha E)F^{-1}$ and $M_2 = CF^{-1}$. Then the left-hand side of (28) with $V = E$ becomes $2\alpha E + \Pi(E) < 0$. So $\left(\sqrt{G}, A, C, \Pi\right)$ is ms-detectable.

Now we prove the ms-stabilizability of solutions to (2). The special case $C = D = S = Z = 0$ of Theorem 14 (i) has been proved in [6] and [22].

Theorem 14. *Suppose (Y, Z) is a pair of upper-lower solutions to (2) and $Y \geq Z$.*
(i) If $\mathcal{R}(Z) > 0$ and $\left(\sqrt{\mathcal{E}(Z)}, A - B\mathcal{K}(Z), C - D\mathcal{K}(Z), \Pi\right)$ is ms-detectable, then Y is ms-stabilizing.
(ii) If $\mathcal{R}(Y) < 0$ and $\left(\sqrt{-\mathcal{E}(Y)}, A - B\mathcal{K}(Y), C - D\mathcal{K}(Y), \Pi\right)$ is ms-detectable, then Z is ms-stabilizing.
In both cases, (2) has a unique solution P in $[Z, Y]$ and P is ms-stabilizing.

Proof. (i) We first assume $S = 0$ and $Z = 0$. The assumptions imply that $G \geq 0$, $R > 0$ and $\left(\sqrt{G}, A, C, \Pi\right)$ is ms-detectable. By (19) with $K = \mathcal{K}(P)$, equation (2) is equivalent to

$$\mathcal{L}(\mathcal{K}(P); P) + \Pi(P) + \mathcal{G}(\mathcal{K}(P)) = 0. \tag{32}$$

Note that $S = 0$ and $\mathcal{G}(\mathcal{K}(P)) = G + \mathcal{K}(P)^T R\mathcal{K}(P)$. So (32) is precisely (27) with $K = \mathcal{K}(P)$. Because Y is an upper solution to (2) and so it also an upper solution to (32), Theorem 13 (ii) implies that $(A - B\mathcal{K}(Y), C - D\mathcal{K}(Y), \Pi)$ is ms-stable; that is, Y is ms-stabilizing. For the general case, consider $P = Y - Z$. Then by (22), equation (2) for P is equivalent to

$$\begin{cases} \widehat{G} + \widehat{A}^T P + P\widehat{A} + \widehat{C}^T P\widehat{C} + \Pi(P) \\ -\left(B^T P + D^T P\widehat{C}\right)^T \left(\widehat{R} + D^T PD\right)^+ \left(B^T P + D^T P\widehat{C}\right) \leq 0, \\ P(t_1) = N - Z(t_1) \geq 0, \end{cases} \tag{33}$$

where $\widehat{A} = A - B\mathcal{K}(Z)$, $\widehat{C} = C - D\mathcal{K}(Z)$, $\widehat{G} = \mathrm{LQ}(Z) + \Pi(Z) \geq 0$ and $\widehat{R} = \mathcal{R}(Z) > 0$. The assumptions imply that (33) has a lower solution 0 with ms-detectable $\left(\sqrt{\widehat{G}}, \widehat{A}, \widehat{C}, \Pi\right)$. This is precisely the case we just proved with A, C, G, R replaced by $\widehat{A}, \widehat{C}, \widehat{G}, \widehat{R}$, respectively. Therefore, P is ms-stabilizing in the sense that $\left(\widehat{A} - B\widehat{\mathcal{K}}(P), \widehat{C} - D\widehat{\mathcal{K}}(P), \Pi\right)$ is ms-stable, where $\widehat{\mathcal{K}}(P) = \left(\widehat{R} + D^T PD\right)^{-1}\left(B^T + D^T P\widehat{C}\right)$. Since $\mathcal{R}(Y) \geq \mathcal{R}(Z) > 0$, it is directly checked that $\widehat{\mathcal{K}}(P) = \mathcal{K}(Y) - \mathcal{K}(Z)$; see [12, proof of Proposition 5]. It follows that

$$(A - B\mathcal{K}(Y), C - D\mathcal{K}(Y), \Pi) = \left(\widehat{A} - B\widehat{\mathcal{K}}(P), \widehat{C} - D\widehat{\mathcal{K}}(P), \Pi\right)$$

is ms-stable; that is, Y is ms-stabilizing. This finishes proof (i). In particular, all solutions to (2) in $[Z, Y]$ are ms-stabilizing and so they must be unique by Proposition 12.

The proof (ii) is similar or it follows from Remark 6.☐

By Remark 3, the ms-detectability condition in Theorem 14 hold if Z is a strict lower solution (i.e., $\mathcal{E}(Z) > 0$) or Y is a strict upper solution ($\mathcal{E}(Y) < 0$). Thus we have the following corollary.

Corollary 15. *Suppose* (Y, Z) *is a pair of upper-lower solutions and* $Y \geq Z$.
(i) If $\mathcal{R}(Z) > 0$ *and* $\mathcal{E}(Z) > 0$, *then* Y *is ms-stabilizing.*
(ii) If $\mathcal{R}(Y) < 0$ *and* $\mathcal{E}(Y) < 0$, *then* Z *is ms-stabilizing.*
In both cases, (2) has a unique solution in $[Z, Y]$ *and it is ms-stabilizing.*

Finally we show that the stabilizing solution to (2) can be approximated by solutions to linear equations.

Theorem 16. *Suppose* (Y, Z) *is a pair of upper and lower solutions to (2) such that* $Y \geq Z$.
(i) Suppose $\mathcal{R}(Z) > 0$ *and* $\left(\sqrt{\mathcal{E}(Z)}, A - B\mathcal{K}(Z), C - D\mathcal{K}(Z), \Pi\right)$ *is ms-detectable. Define* $H_1 = Y$ *and* H_{i+1} *be the unique solution to*

$$\mathcal{L}(\mathcal{K}(H_i); P) + \mathcal{G}(\mathcal{K}(H_i)) + \Pi(P) = 0 \tag{34}$$

for $i \geq 1$. *Then* $H_1 \geq H_2 \geq \cdots \geq Z$ *and* $H = \lim\limits_{i\to\infty} H_i$ *is the unique ms-stabilizing solution to (2) in* $[Z, Y]$.
(ii) Suppose $\mathcal{R}(Y) < 0$ *and* $\left(\sqrt{-\mathcal{E}(Y)}, A - B\mathcal{K}(Y), C - D\mathcal{K}(Y), \Pi\right)$ *is ms-detectable. Define* $H_1 = Z$ *and* H_{i+1} *be the solution to (34) for* $i \geq 1$. *Then* $H_1 \leq H_2 \leq \cdots \leq Y$ *and* $H = \lim\limits_{i\to\infty} H_i$ *is the unique ms stabilizing solution to (2) in* $[Z, Y]$.

Proof. (i) We first show by induction that for $i \geq 1$, H_i is an ms-stabilizing upper solution to (2) and $H_i \geq Z$. For $i = 1$, we have $H_1 = Y \geq Z$ and Y is an upper solution to (2) by assumption. By Theorem 14(i), Y is ms-stabilizing. Suppose now that H_i is ms-stabilizing upper solution to (2) and $H_i \geq Z$ and show that H_{i+1} is an ms-stabilizing upper solution to (2) and $H_{i+1} \geq Z$. Using that Z is a lower solution with $\mathcal{R}(Z) \geq 0$ and from (20.1) with $P = Z$ and $K = \mathcal{K}(H_i)$, we have

$$0 \leq \text{LQ}(Z) + \Pi(Z) \leq \mathcal{L}(\mathcal{K}(H_i); Z) + \mathcal{G}(\mathcal{K}(H_i)) + \Pi(Z).$$

This shows that Z is a lower solution to (34). Since H_i is ms-stabilizing and H_{i+1} is a solution to (34), by Theorem 10 (ii), $H_{i+1} \geq Z$. Consequently, $\mathcal{R}(H_{i+1}) \geq \mathcal{R}(Z) > 0$. By (20.1) with $P = H_{i+1}$ and $K = \mathcal{K}(H_i)$, we have

$$\text{LQ}(H_{i+1}) + \Pi(H_{i+1}) \leq \mathcal{L}(\mathcal{K}(H_i); H_{i+1}) + \mathcal{G}(\mathcal{K}(H_i)) + \Pi(H_{i+1}) = 0,$$

where the last equation is just (34) for H_{i+1}. So H_{i+1} is an upper solution to (2). By Theorem 14, H_{i+1} must be ms-stabilizing.

Next we show that $H_i \geq H_{i+1}$. Using that H_i is an upper solution to (2) and from (19) with $P = H_i$, we have

$$0 \geq \text{LQ}(H_i) + \Pi(H_i) = \mathcal{L}(\mathcal{K}(H_i); H_i) + \mathcal{G}(\mathcal{K}(H_i)) + \Pi(H_i).$$

This shows that H_i is an upper solutions to (34). By Theorem 10 (ii) again, $H_i \geq H_{i+1}$.

Now it is clear that the limit $H = \lim_{i\to\infty} H_i$ exists and satisfies (2). By Theorem 14, H is stabilizing. The proof of (ii) is similar or it follows from Remark 6.☐

Appendix of Notation

We collect here the notations frequently used in this paper.

$\mathbb{S}^n$ = the set of all real symmetric $n \times n$ matrices

$L^\infty(I, \mathbb{X}) =$ the space of all bounded and measurable functions from I to $\mathbb{X}$.

$L^{1,\infty}(I, \mathbb{X}) = \{P \in L^\infty(I, \mathbb{X}), P' \in L^\infty(I, \mathbb{X})\}$

$\Pi : \mathbb{S}^n \to \mathbb{S}^n$, $\Pi \in L^\infty(\mathbb{S}^n, \mathbb{S}^n)$ is linear and $\Pi \geq 0$

$A, C \in \mathbb{R}^{n\times n}$

$B, D,\ S^T \in \mathbb{R}^{n\times k}$

$R \in \mathbb{S}^k$

$G, N \in \mathbb{S}^n$

$P, Y, Z \in \mathbb{S}^n$ or $L^{1,\infty}(I, \mathbb{S}^n)$

$\mathcal{E}(P) = P' + \mathrm{LQ}(P) + \Pi(P)$ or $\mathcal{E}(P) = \mathrm{LQ}(P) + \Pi(P)$ (l.h.s. of (1) or (2))

$\mathrm{L}(P) = A^T P + PA + C^T PC$

$\mathrm{Q}(P) = (B^T P + D^T PC + S)^T (R + D^T PD)^{-1} (B^T P + D^T PC + S)$

$\mathrm{LQ}(P) = G + \mathrm{L}(P) - \mathrm{Q}(P)$

$\mathcal{R}(P) = R + D^T PD$; note $\mathcal{R}(0) = R$

$\mathcal{S}(P) = B^T P + D^T PC + S$; note $\mathcal{S}(0) = S$

$\mathcal{K}(P) = \mathcal{R}(P)^+ \mathcal{S}(P)$, where $\mathcal{R}(P)^+$ is the pseudoinverse of $\mathcal{R}(P)$

$\mathbb{K}(P) = \{K \in \mathbb{R}^{k\times n} : \mathcal{S}(P) = \mathcal{R}(P)K\}$-the set of feedback matrices associated with P

$\mathcal{G}(K) = K^T RK - K^T S - S^T K + G$; note $\mathcal{G}(0) = G$

$\mathcal{L}(K; P) = (A - BK)^T P + P(A - BK) + (C - DK)^T P(C - DK)$

Note that $\mathcal{L}(0; P) = \mathrm{L}(P)$.

References

[1] M. Ait Rami and X.Y. Zhou, Linear matrix inequalities, Riccati equations, indefinite stochastic linear quadratic controls, *IEEE, Transactions on Automatic Control*, Vol. 45, No.6 (2000) 1131-1143.

[2] T. Basar and P. Bernhard, H^∞-Optimal Control and Related Minimax Design Problems, Birkhauser, 1995.

[3] S. Bittanti, A.J. Laub, and J.C. Willems, The Riccati Equations, Springer, 1991.

[4] R. W. Brockett, Finite Dimensional Linear Systems, J. Wiley, New York, 1970.

[5] L. Faybusovich, Matrix Riccati inequality: existence of solutions. *Systems, Control Lett*. 9 (1987), no. 1, 59-64.

[6] M.D. Fragoso, O.L.V. Costa and C.E. de Souza, A new approach to linearly perturbed Riccati equations arising in stochastic control, *Applied Mathematics & Optimization*, 37 (1998) 99-126.

[7] G. Freiling and G. Jank, Existence and comparison theorems for algebraic Riccati equations and Riccati differential and difference equations, *J. Dynam. Contr. System*, 2 (1996) 529-547.

[8] I. Gohberg, P. Lancaster, and L. Rodman, On Hermitian solutions of the symmetric algebraic Riccati equations, *SIAM J. Contr. Optimiz.* 24 (1986), 1323-1334.

[9] V. Ionescu, C. Oara and M. Weiss, Generalized Riccati Theory and Robust Control, A Popov Approach, John Wiley & Sons, 1999.

[10] P. Lancaster and L. Rodman, The Algebraic Riccati Equations, Oxford University Press, Oxford, 1995.

[11] M. McAsey and L. Mou, Riccati algebraic and differential equations from stochastic differential games. In preparation, 2001.

[12] L. Mou, Upper-lower solution method for differential Riccati equations from stochastic LQR problems. Preprint, 2001.

[13] L. Mou and S. R. Liberty, Estimation of maximal existence intervals for solutions to a Riccati equation via an upper-lower solution method. To appear on the *Proceedings of Thirty-Nineth Allerton Conference on Communication, Control, and Computing*, 2001.

[14] R. Penrose, A generalized inverse of matrices, *Proc. Cambridge Philos. Soc.*, 52 (1955), 17-19.

[15] A.C.M. Ran and R. Vreugdenhil, Existence and comparison theorems for algebraic Riccati equations for continuous- and discrete-time systems, *Linear Algebra Appl*. 99 (1988), 63-83.

[16] W.T. Reid, Riccati Differential Equations. Academic Press, New York and London, 1972.

[17] C. Scherer, The solution set of the algebraic Riccati equation and the algebraic Riccati inequality. *Linear Algebra and its Applications*, 153 (1991) 99-122.

[18] D.D. Sworder, Feedback control of a class of linear systems with jump parameters, *IEEE Trans. Automat. Contr.*, AC-14 (1969) 9-14.

[19] J. C. Willems, Least squares stationary optimal control and the algebraic Riccati equation. *IEEE Trans. Automatic Control,* AC-16 (1971), 621--634.

[20] C. W. Wimmer, Monotonicity of maximal solutions of algebraic Riccati equations, *Syst. Contr. Lett*. 5 (1985), 317-319.

[21] H.K. Wimmer, A Galois correspondence between sets of semidefinite solutions of continuous-time algebraic Riccati equations, *Linear Algebra and its Applications*, 205-206 (1994) 1253-1270.

[22] W.M. Wonham, On a matrix Riccati equation of stochastic control, *SIAM J. Control*, Vol. 6, No.4 (1968) 681-697.

[23] D.D. Yao, S. Zhang and X.Y. Zhou, Stochastic linear-quadratic control via semidefinite programming, *SIAM J. Control and Optimizations* 40: 813-823 (2001).

[24] J. Yong and X.Y. Zhou, Stochastic Controls, Hamiltonian Systems and HJB Equations, Springer 1999.

[25] K. Zhou with J. Doyle and K. Glover, Robust Optimal Control, Prentice Hall 1995.