

On Azumaya Commutator Hopf Galois Extensions

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ABSTRACT. Let H be a finite dimensional Hopf algebra over a field k , H^* the dual Hopf algebra of H , B a left H -module algebra, and $V_B(B^H)$ the commutator subring of B^H in B . It is shown that $V_B(B^H)$ is an H^* -Galois extension and an Azumaya algebra if and only if the smash product $B\#H$ is a Hirata separable extension of B and B is a Hirata separable extension of B^H .

1 Introduction.

Let H be a finite dimensional Hopf algebra over a field k , H^* the dual Hopf algebra of H , B a left module algebra over H with center C , B^H the H -invariants of B , and $V_B(B^H)$ the commutator subring of B^H in B . In [6], B is called a commutator Hopf Galois extension of B^H if $V_B(B^H)$ is a right H^* -Galois extension of C^H . It was shown that $V_B(B^H)$ is an H^* -Galois extension of C^H if and only if $B \cong B^H \otimes_{C^H} V_B(B^H)$ and $V_B(B^H)$ is a finitely generated and projective C^H -module ([3], Lemma 2.8). More characterizations were given in terms of the smash products $B\#H$ and $V_B(B^H)\#H$ ([6], Theorem 3.1). We note that the class of commutator Hopf Galois extensions contains the class of Hopf Galois Azumaya extensions as studied in [4]. The purpose of the present paper is to study another class of commutator Hopf Galois extensions which are called Azumaya commutator Hopf Galois extensions, that is, B is an Azumaya commutator Hopf Galois extension of B^H . Several characterizations are given in terms of Hirata separable extensions, and expressions are

also obtained for B and $B\#H$. Moreover, a relationship between an Azumaya commutator Hopf Galois extension and an Hopf Galois Azumaya extension is derived.

2 Basic Definitions and Notations.

Throughout, H denotes a finite dimensional Hopf algebra over a field k with comultiplication Δ and counit ε , H^* the dual Hopf algebra of H , B a left H -module algebra, C the center of B , $B^H = \{b \in B \mid hb = \varepsilon(h)b \text{ for all } h \in H\}$ which is called the H -invariants of B , and $B\#H$ the smash product of B with H where $B\#H = B \otimes_k H$ such that for all $b\#h$ and $b'\#h'$ in $B\#H$, $(b\#h)(b'\#h') = \sum b(h_1b')\#h_2h'$ where $\Delta(h) = \sum h_1 \otimes h_2$. B is called a right H^* -Galois extension of B^H if B is a right H^* -comodule algebra with structure map $\rho : B \rightarrow B \otimes_k H^*$ such that $\beta : B \otimes_{B^H} B \rightarrow B \otimes_k H^*$ is a bijection where $\beta(a \otimes b) = (a \otimes 1)\rho(b)$.

For a subring A of B with the same identity 1, we denote the commutator subring of A in B by $V_B(A)$. we call B a separable extension of A if there exist $\{a_i, b_i \text{ in } B, i = 1, 2, \dots, m \text{ for some integer } m\}$ such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where \otimes is over A . An Azumaya algebra is a separable extension of its center. A ring B is called a Hirata separable extension of A if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B -bimodule. A right H^* -Galois extension B is called a right H^* -Galois Azumaya extension if B is separable over B^H which is an Azumaya algebra over C^H , a commutator right H^* -Galois extension if $V_B(B^H)$ is a right H^* -Galois extension of C^H , and an Azumaya commutator right H^* -Galois extension if $V_B(B^H)$ is an Azumaya and a right H^* -Galois extension of C^H . Throughout, an H^* -Galois extension means a right H^* -Galois extension unless it is stated otherwise.

3. Main results.

In this section, several characterizations of an Azumaya commutator Hopf Galois extension are given in terms of the concept of Hirata separability, expressions of B and $B\#H$ are also obtained, and then a relationship to Hopf Galois Azumaya extensions is derived.

Theorem 3.1. *Let B be a left H -module algebra. Then the following statements are equivalent:*

- (1) B is an Azumaya commutator H^* -Galois extension of B^H ,
- (2) $B\#H$ is a Hirata separable extension of B and B is a Hirata separable extension of B^H ,
- (3) $V_B(B^H)$ is an Azumaya C^H -algebra and B^H satisfies the double centralizer property in B (that is, $V_B(V_B(B^H)) = B^H$), and
- (4) $V_B(B^H)$ is an H^* -Galois extension of C^H and B is a Hirata separable extension of B^H .

Proof. (1) \implies (2) By hypothesis, $V_B(B^H)$ is an H^* -Galois extension of C^H . so $B \cong B^H \otimes_{C^H} V_B(B^H)$ ([3], Lemma 2.8). Moreover, since $V_B(B^H)$ is an Azumaya C^H -algebra, B ($\cong B^H \otimes_{C^H} V_B(B^H)$) is a Hirata separable extension of B^H . Again, $B \cong B^H \otimes_{C^H} V_B(B^H)$ where $V_B(B^H)$ is an H^* -Galois extension of C^H , so B is an H^* -Galois extension of B^H with a same Galois system as $V_B(B^H)$ ([4], Theorem 5.1). Thus $B\#H \cong \text{Hom}_{B^H}(B, B)$ ([4], Theorem 2.2) such that B is a Hirata separable extension of B^H and $V_B(B^H)$ is an Azumaya C^H -algebra. Therefore $B\#H$ ($\cong \text{Hom}_{B^H}(B, B)$) is a Hirata separable extension of B ([5], Proposition 12).

(2) \implies (1) Since $B\#H$ is a Hirata separable extension of B , B , being a generator of itself, is a generator of $B\#H$ ([2], Lemma). Hence B is an H^* -Galois extension of B^H ([4], Theorem 2.2). Thus $B\#H \cong \text{Hom}_{B^H}(B, B)$. By hypothesis, $B\#H$ is a Hirata separable extension of B and B is a Hirata separable extension of B^H . Hence $V_B(B^H)$ is an Azumaya

C -algebra ([5], Proposition 12). On the other hand, B is an H^* -Galois extension and a Hirata separable extension of B^H , so $V_B(V_B(B^H)) = B^H$ ([7], Lemma 2.5 and Theorem 2.6). Thus $C = C^H$ and $B \cong B^H \otimes_{C^H} V_B(B^H)$ ([1], Corollary 3.6, page 54). This implies that $V_B(B^H)$ is an Azumaya C^H -algebra and an H^* -Galois extension of C^H ([3], Lemma 2.8).

(1) \implies (3) Since $V_B(B^H)$ is an H^* -Galois extension of C^H , B is an H^* -Galois extension of B^H with a same H^* -Galois system as $V_B(B^H)$. But statement (1) implies statement (2), so B is a Hirata separable extension of B^H . Thus B^H satisfies the double centralizer property in B ; that is, $V_B(V_B(B^H)) = B^H$ ([7], Lemma 2.5 and Theorem 2.6).

(3) \implies (1) Since $V_B(B^H)$ is an Azumaya C^H -algebra, $B \cong V_B(B^H) \otimes_{C^H} V_B(V_B(B^H))$ ([1], Corollary 3.6, page 54). By hypothesis, $V_B(V_B(B^H)) = B^H$, so $B \cong V_B(B^H) \otimes_{C^H} B^H$. Thus $V_B(B^H)$ is an H^* -Galois extension of C^H ([3], Lemma 2.8).

(1) \implies (4) Since (1) \implies (2), B is a Hirata separable extension of B^H . Thus (4) holds.

(4) \implies (2) It suffices to show that $B\#H$ is a Hirata separable extension of B . By hypothesis, $V_B(B^H)$ is an H^* -Galois extension of C^H . Hence $B \cong B^H \otimes_{C^H} V_B(B^H)$ such that $V_B(B^H)$ is a finitely generated and projective C^H -module ([3], Lemma 2.8). Noting that C^H is a commutative ring, we have that $V_B(B^H)$ is a progenerator of C^H . Thus $\text{Hom}_{C^H}(V_B(B^H), V_B(B^H))$ is an Azumaya C^H -algebra ([1], Proposition 4.1, page 56). But $\text{Hom}_{C^H}(V_B(B^H), V_B(B^H)) \cong (V_B(B^H))^* \otimes_{C^H} V_B(B^H)$ for $V_B(B^H)$ is a finitely generated and projective C^H -module. Hence $\text{Hom}_{C^H}(V_B(B^H), V_B(B^H))$ is a finitely generated and projective $V_B(B^H)$ -module; and so $\text{Hom}_{C^H}(V_B(B^H), V_B(B^H))$ is a Hirata separable extension of $V_B(B^H)$ ([2], Theorem 1). Moreover, $V_B(B^H)$ is an H^* -Galois extension of C^H , so $V_B(B^H)\#H \cong \text{Hom}_{C^H}(V_B(B^H), V_B(B^H))$. Therefore $V_B(B^H)\#H$ is a Hirata separable extension of $V_B(B^H)$; and so $B\#H (\cong B^H \otimes_{C^H} (V_B(B^H)\#H))$ is a Hirata separable extension of $B (\cong B^H \otimes_{C^H} V_B(B^H))$.

By the proof of theorem 3.1, we have an expression of $B\#H$.

Corollary 3.2. *If B is an Azumaya commutator H^* -Galois extension of B^H , then $B\#H \cong B^H \otimes_{C^H} (V_B(B^H)\#H)$ such that $V_B(B^H)\#H$ is an Azumaya C^H -algebra.*

The converse of Corollary 3.2 holds when $V_B(B^H)$ is an Azumaya C^H -algebra.

Theorem 3.3. *If $B\#H \cong B^H \otimes_{C^H} (V_B(B^H)\#H)$ such that $V_B(B^H)$ and $V_B(B^H)\#H$ are Azumaya C^H -algebras, then B is an Azumaya commutator H^* -Galois extension of B^H*

Proof. Since $B\#H \cong B^H \otimes_{C^H} (V_B(B^H)\#H)$, $B \cong B^H \otimes_{C^H} V_B(B^H)$. But $V_B(B^H)$ is an Azumaya C^H -algebra, so $B^H \otimes_{C^H} V_B(B^H)$ is a Hirata separable extension of B^H . Hence B is a Hirata separable extension of B^H . Moreover, by hypothesis, $V_B(B^H)\#H$ is an Azumaya C^H -algebra, so it is a Hirata separable extension of $V_B(B^H)$ ([2], Theorem 1). But then $B^H \otimes_{C^H} (V_B(B^H)\#H)$ is a Hirata separable extension of $B^H \otimes_{C^H} V_B(B^H)$. Thus $B\#H$ is a Hirata separable extension of B . Therefore B is an Azumaya commutator H^* -Galois extension of B^H by Theorem 3.1.

Next, we give a relationship between an Azumaya commutator H^* -Galois extension and an H^* -Galois Azumaya extension.

Theorem 3.4. *Let B be an Azumaya commutator H^* -Galois extension of B^H . Then B is an H^* -Galois Azumaya extension of B^H if and only if B^H is separable over C^H .*

Proof. (\implies) Since B is an H^* -Galois Azumaya extension of B^H , B^H is an Azumaya algebra over C^H ([4], Theorem 3.4).

(\impliedby) Since B is an Azumaya commutator H^* -Galois extension of B^H , $V_B(B^H)$ is an H^* -Galois extension of C^H . Hence $B \cong B^H \otimes_{C^H} V_B(B^H)$ ([3], Lemma 2.8) and B is an H^* -Galois extension of B^H . Moreover, $V_B(B^H)$ is an Azumaya C^H -algebra, so B ($\cong B^H \otimes_{C^H} V_B(B^H)$) is separable over B^H and $C = C^H$. By hypothesis, B^H is separable

over C^H , so B^H is an Azumaya C^H -algebra. Thus B is an H^* -Galois Azumaya extension of B^H ([4], Theorem 3.4).

By the proof of Theorem 3.4, the following is immediate.

Theorem 3.5. *Let B be an H^* -Galois Azumaya extension of B^H . Then B is an Azumaya commutator H^* -Galois extension of B^H if and only if $C = C^H$.*

ACKNOWLEDGEMENTS. This paper was written under the support of a Caterpillar Fellowship at Bradley University. The author would like to thank Caterpillar Inc. for the support.

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