On Certain Correspondences of Separable Extensions in a Commutator Galois Extension

Lianyong Xue
Department of Mathematics
Bradley University
Peoria, Illinois 61625, USA
E-Mail: lxue@bradley.edu

Abstract

Let B be a commutator Galois extension of B^G with Galois group G such that C^G is a direct summand of B^G where C is the center of B. Then it is shown that there exists a one-to-one correspondence between the set of separable extensions A of B^G in B such that A is a direct summand of B as an A-bimodule and the set of separable C^G-subalgebras of $V_B(B^G)$ where $V_B(B^G)$ is the commutator subring of B^G in B. This generalizes F. DeMeyer's result for DeMeyer-Kanzaki Galois extensions and derives a classification for Azumaya Galois extensions satisfying the fundamental theorem. More correspondences for a commutator Galois extension are also given.

1. Introduction

Let $F \subset K$ be a finite field Galois extension with Galois group G. It is well known that the fundamental theorem holds for $F \subset K$, that is, the map $\alpha : H \longrightarrow K^H$ for a subgroup H of G is a one-to-one correspondence between the set of subgroups of G and

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the set of separable subfields of K over F. In [3], S.U. Chase, D.K. Harrison, and A. Rosenberg extended this fact to finite indecomposable commutative ring Galois extensions (with no idempotents but 0 and 1). In [4], for a noncommutative Galois extension B of B^G with Galois group G such that B is an Azumaya algebra over C where C is a commutative Galois algebra over C^G with Galois group $G|_C \cong G$, there exists a one-to-one correspondence between the set of separable extensions of B^G in B and the set of separable C^G-subalgebras of C. Such a Galois extension is called a DeMeyer-Kanzaki Galois extension. Noting that a DeMeyer-Kanzaki Galois extension B of B^G with Galois group G is a center Galois extension of B^G such that B^G is an Azumaya C^G-algebra; and so any separable extensions A of B^G in B is a direct summand of B as an A-bimodule. In the present paper, assuming the weaker conditions that $V_B(B^G)$ is a Galois extension of $(V_B(B^G))^G$ with Galois group $G|_{V_B(B^G)} \cong G$ and C^G is a direct summand of B^G, we show that there exists a one-to-one correspondence between the set of separable extensions A of B^G in B such that A is a direct summand of B as an A-bimodule and the set of separable C^G-subalgebras of $V_B(B^G)$. This generalizes the DeMeyer's result and derives a classification for Azumaya Galois extensions satisfying the fundamental theorem.

2. Definitions and Notations

Throughout this paper, B will represent a ring with 1, C the center of B, G a finite automorphism group of B, B^G the set of elements in B fixed under each element in G, $B*G$ the skew group ring of G over B, that is, $B*G$ is the free left B-module in which the multiplication is given by $gb = g(b)g$ for $b \in B$ and $g \in G$, and $\overline{G}$ the inner automorphism group of $B * G$ induced by G, that is, $\overline{g}(x) = gxg^{-1}$ for each $x \in B * G$ and $g \in G$. We note that $\overline{G}$ restricted to B is G.

Let A be a subring of a ring B with the same identity 1. We denote $V_B(A)$ the commutator subring of A in B and $G(A)$ the subgroup $\{g \in G \,|\, g(a) = a$ for all $a \in A\}$ of G. We call B a separable extension of A if there exist $\{a_i, b_i$ in B, $i = 1, 2, ..., k$ for some integer $k\}$ such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where $\otimes$ is over A. An Azumaya algebra is a separable extension of its center. A ring B is called a H-separable extension of A (i.e., Hirata separable) if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B-bimodule. We call B a Galois extension of B^G with Galois group G if there exist elements $\{a_i, b_i$ in B, $i = 1, 2, ..., m$ for some integer $m\}$ such that $\sum_{i=1}^m a_i g(b_i) = \delta_{1,g}$ for each $g \in G$. Such a set $\{a_i, b_i\}$ is called a G-Galois system for B. We call B a center Galois extension with Galois group G if C is a Galois algebra over C^G with Galois group $G|_C \cong G$, and a commutator Galois extension of B^G with Galois group G if $V_B(B^G)$ is a Galois extension of $(V_B(B^G))^G$ with Galois group $G|_{V_B(B^G)} \cong G$. A Galois extension B of B^G with Galois group G is called a DeMeyer-Kanzaki Galois extension if B is an Azumaya algebra over C and a center Galois extension of B^G with Galois group G. An Azumaya Galois extension B of B^G with Galois group G is a Galois extension B of B^G such that B^G is an Azumaya C^G-algebra. A ring B is called indecomposable if it contains no central idempotents but 0 and 1. We call a Galois extension B with Galois group G satisfying the fundamental theorem if $\alpha : H \longrightarrow B^H$ for a subgroup H of G is a one-to-one correspondence between the set of subgroups of G and the set of separable subextensions of B^G in B. As in [14], we denote $J_g = \{b \in B \,|\, bx = g(x)b$ for each $x \in B\}$ for a $g \in G$.

3. Main Results

Keeping the definitions and notations in section 2, in this section, we shall generalize the one-to-one correspondence given in Lemma 2 in [4] and derives a classification for Azumaya Galois extensions satisfying the fundamental theorem. We begin with some properties of a commutator Galois extension.

LEMMA 3.1.

Let B be a commutator Galois extension of B^G with Galois group G. Then B is finitely generated projective as a B^G-bimodule.

Proof. Since B is a commutator Galois extension of B^G with Galois group G, B has a Galois system $\{b_i, d_i \in V_B(B^G)\,, i = 1, 2, ..., n$ for some integer $n\}$, that is, $\sum_{i=1}^n b_i g(d_i) = \delta_{1,g}$ for each $g \in G$. Let $f_i : B \longrightarrow B^G$ defined by $f_i(b) = \sum_{g\in G} g(d_i b)$ for all $b \in B$, $i = 1, 2, \cdots n$. Then, it is easy to check that f_i is a B^G-bimodule homomorphism. Moreover, for any $b \in B$,

$$\sum_{i=1}^m b_i f_i(b) = \sum_{i=1}^m b_i \sum_{g\in G} g(d_i b) = \sum_{g\in G}\sum_{i=1}^m b_i g(d_i b) = \sum_{g\in G}\big(\sum_{i=1}^m b_i g(d_i)\big) g(b) = b.$$

Now we define a B^G-bimodule map

$$\alpha : (B^G)^n \longrightarrow B$$

by $\alpha(a_1, a_2, ..., a_n) = \sum_{i=1}^n b_i a_i$ for all $(a_1, a_2, ..., a_n) \in (B^G)^n$. Since $b_i \in V_B(B^G)\,, i = 1, 2, ..., n$, it is clear that α is a B^G-bimodule homomorphism. Moreover, for any $b \in B$,

$$\alpha(f_1(b), f_2(b), ..., f_n(b)) = \sum_{i=1}^n b_i f_i(b) = b.$$

Thus α is surjective. Next, we claim that α has a splitting B^G-bimodule homomorphism. In fact, let $\beta : B \longrightarrow (B^G)^n$ by $\beta(b) = (f_1(b), f_2(b), ..., f_n(b))$. Then β is a B^G-bimodule homomorphism since each f_i is a B^G-bimodule homomorphism. Moreover, for any $b \in B$,

$$\alpha\beta(b) = \alpha(f_1(b), f_2(b), ..., f_n(b)) = \sum_{i=1}^n b_i f_i(b) = b,$$

that is, $\alpha\beta = 1_B$. This shows that β is the splitting B^G-bimodule homomorphism of α, and so B is finitely generated projective as a B^G-bimodule.

LEMMA 3.2.

Let B be a commutator Galois extension of B^G with Galois group G. Then

(1) The center of B^G is C^G, the center of $V_B(B^G)$ is C, and $V_{B*G}(B^G) = V_B(B^G) * G$.

(2) $B \cong B^G \otimes_{C^G} V_B(B^G)$ by the multiplication map and $V_B(B^G)$ is finitely generated projective over C^G.

PROOF. (1) Since B is a commutator Galois extension of B^G with Galois group G, $V_B(B^G)$ is a Galois extension of $(V_B(B^G)^G$ with Galois group $G|_{V_B(B^G)} \cong G$. Noting that $V_B(B^G) \subset B^G \cdot V_B(B^G) \subset B$, we have that both $B^G \cdot V_B(B^G)$ and B are Galois extensions of B^G with Galois group $G|_{B^G \cdot V_B(B^G)} \cong G$. Thus $B = B^G \cdot V_B(B^G)$. Then it is clear that the center of B^G is C^G, the center of $V_B(B^G)$ is C, and $V_{B*G}(B^G) = V_B(B^G) * G$.

(2) Since B is a commutator Galois extension of B^G with Galois group G, B is finitely generated projective as a B^G-bimodule by Lemma 3.1. Noting that the center of B^G is C^G by part (1), we have that $B \cong B^G \otimes_{C^G} V_B(B^G)$ as B^G-bimodules by the multiplication map $a \otimes b \longrightarrow ab$ for $a \otimes b \in B^G \otimes_{C^G} V_B(B^G)$ and $V_B(B^G)$ is finitely generated projective over C^G ([5], Proposition 5.2). It is easy to check that the isomorphism $B \cong B^G \otimes_{C^G} V_B(B^G)$ by the multiplication map is also a ring isomorphism.

LEMMA 3.3.

Let B be a commutator Galois extension of B^G with Galois group G. Then for any separable extension A of B^G in B such that A is a direct summand of B as an A-bimodule, $A = B^G \cdot V_A(B^G) \cong B^G \otimes_{C^G} V_A(B^G)$ by the multiplication map.

PROOF. Since B is a commutator Galois extension of B^G with Galois group G, B is finitely generated projective as a B^G-bimodule by Lemma 3.1. Let A be a separable extension of B^G in B such that A is a direct summand of B as an A-bimodule. Then A is also a direct summand of B as a B^G-bimodule. Hence A is finitely generated projective as a B^G-bimodule because B is finitely generated projective as a B^G-bimodule. Since the center of B^G is C^G by Lemma 3.2, we have that $A \cong B^G \otimes_{C^G} V_A(B^G)$ by the multiplication map ([5], Theorem 5.2).

LEMMA 3.4.

Let B be an Azumaya Galois extension of B^G with Galois group G. Then (1) any separable extension A of B^G in B is a direct summand of B as an A-bimodule and (2) B^H is a separable extension of B^G for each subgroup H of G.

PROOF. (1) Since B is an Azumaya Galois extension of B^G with Galois group G, B^G and $B * G$ are Azumaya C^G-algebras ([2], Theorem 1). Let A be a separable extension of B^G in B. Then A is a separable algebra over C^G by the transitivity of separable extensions. Hence A is a direct summand of $B * G$ as an A-bimodule ([9], Proposition 1.5). Noting that $A \subset B$, we have that A is a direct summand of B as an A-bimodule.

(2) Since B is an Azumaya Galois extension of B^G with Galois group G, B^G is a separable algebra over C^G. Hence B^H is a separable algebra over C^G for each subgroup H of G ([8], Proposition 3.1). Thus B^H is a separable extension of B^G for each subgroup H of G.

COROLLARY 3.5.

Let B be a Galois algebra over a commutative ring R with Galois group G. Then (1) any separable R-subalgebra A of B is a direct summand of B as an A-bimodule and (2) B^H is a separable R-algebra for each subgroup H of G.

THEOREM 3.6.

Let B be a commutator Galois extension of B^G with Galois group G such that C^G is a direct summand of B^G. Then $\alpha : S \longrightarrow B^G \cdot S$ is a one-to-one correspondence between the set of separable C^G-subalgebras of $V_B(B^G)$ and the set of separable extensions

A of B^G in B such that A is a direct summand of B as an A-bimodule with inverse map $\alpha^{-1} : A \longrightarrow V_A(B^G)$.

PROOF. Since B is a commutator Galois extension of B^G with Galois group G, $B = B^G \cdot V_B(B^G) \cong B^G \otimes_{C^G} V_B(B^G)$ by Lemma 3.2 and $V_B(B^G)$ is a Galois algebra over C^G with Galois group $G|_{V_B(B^G)} \cong G$ ([13], Lemma 3.1). Let S be a separable C^G-subalgebra of $V_B(B^G)$. Then $B^G \cdot S \cong B^G \otimes_{C^G} S$ which is a separable extension of B^G in B. Moreover, S is a direct summand of $V_B(B^G)$ as a S-bimodule by Corollary 3.5-(1). Hence $B^G \otimes_{C^G} S$ is a direct summand of $B^G \otimes_{C^G} V_B(B^G)$ as a $B^G \otimes_{C^G} S$-bimodule. This shows that $\alpha : S \longrightarrow B^G \cdot S$ is well defined from the set of separable C^G-subalgebras of $V_B(B^G)$ to the set of separable extensions A of B^G in B such that A is a direct summand of B as an A-bimodule. We claim that α is one-to-one and onto. In fact, let S_1 and S_2 be two separable C^G-subalgebras of $V_B(B^G)$ such that $\alpha(S_1) = \alpha(S_2)$, that is, $B^G \cdot S_1 = B^G \cdot S_2$. Then, noting that $B = B^G \cdot V_B(B^G) \cong B^G \otimes_{C^G} V_B(B^G)$, we have that $S_1 = V_{B^G \cdot S_1}(B^G) = V_{B^G \cdot S_2}(B^G) = S_2$. This shows that α is one-to-one. On the other hand, let A be a separable extension of B^G in B such that A is a direct summand of B as an A-bimodule. Since B is a commutator Galois extension of B^G with Galois group G, $A = B^G \cdot V_A(B^G) \cong B^G \otimes_{C^G} V_A(B^G)$ by Lemma 3.3. Moreover, since C^G is a direct summand of B^G, that $B^G \otimes_{C^G} V_A(B^G)$ $(\cong A)$ is separable over B^G implies that $V_A(B^G)$ is separable over C^G ([6], Corollary 2.12), and so $\alpha(V_A(B^G)) = A$. This proves that α is onto, and so α is a one-to-one correspondence between the set of separable C^G-subalgebras of $V_B(B^G)$ and the set of separable extensions A of B^G in B such that A is a direct summand of B as an A-bimodule with inverse map $\alpha^{-1} : A \longrightarrow V_A(B^G)$.

Next, we shall apply Theorem 3.6 to Azumaya Galois extensions and show a classification for Azumaya Galois extensions satisfying the fundamental theorem.

THEOREM 3.7.

Let B be an Azumaya Galois extension of B^G with Galois group G. Then B satisfies the fundamental theorem if and only if so does $V_B(B^G)$.

PROOF. Since B is an Azumaya Galois extension of B^G with Galois group G, B is a commutator Galois extension of B^G with Galois group G ([1], Theorem 2). Since B is an Azumaya Galois extension of B^G with Galois group G again, B^G is an Azumaya algebra over C^G. Hence C^G is a direct summand of B^G. Thus, by Theorem 3.6, $\alpha : S \longrightarrow B^G \cdot S$ is a one-to-one correspondence between the set of separable extensions A of B^G in B such that A is a direct summand of B as an A-bimodule and the set of separable C^G-subalgebras of $V_B(B^G)$ with inverse map $\alpha^{-1} : A \longrightarrow V_A(B^G)$. Moreover, by Lemma 3.4-(1), any separable extension A of B^G in B is a direct summand of B as an A-bimodule. Thus $\alpha : S \longrightarrow B^G \cdot S$ is a one-to-one correspondence between the set of separable extensions of B^G in B and the set of separable C^G-subalgebras of $V_B(B^G)$. Assume that B satisfies the fundamental theorem. Then $\beta : H \longrightarrow B^H$ is a one-to-one correspondence between the set of subgroups of G and the set of separable extensions of B^G in B. Hence $\alpha^{-1}\beta : H \longrightarrow B^H \longrightarrow V_{B^H}(B^G)$ is a one-to-one correspondence between the set of subgroups of G and the set of separable C^G-subalgebras of $V_B(B^G)$. Moreover, since $V_{B^H}(B^G) = V_B(B^G) \cap B^H = (V_B(B^G))^H$, $\alpha^{-1}\beta : H \longrightarrow (V_B(B^G))^H$ is the Galois correspondence. Conversely, assume $V_B(B^G)$ satisfies the fundamental theorem. Then $\beta' : H \longrightarrow (V_B(B^G))^H$ is a one-to-one correspondence between the set of subgroups of G and the set of separable C^G-subalgebras of $V_B(B^G)$. Hence $\alpha\beta' : H \longrightarrow V_{B^H}(B^G) \longrightarrow B^G \cdot V_{B^H}(B^G)$ is a one-to-one correspondence between the set of subgroups of G and the set of separable extensions of B^G in B. Moreover, since B is an Azumaya Galois extension of B^G with Galois group G, B^H is a separable extension of B^G for each subgroup H of G by Lemma 3.4-(2). Hence $B^H = B^G \cdot V_{B^H}(B^G)$ by Lemma 3.4-(1) and Lemma 3.3. Therefore $\alpha\beta' : H \longrightarrow B^H$ is the Galois correspondence.

Combining Theorem 4.7 in [14] and Corollary 3.5-(2), we have the following proposition.

PROPOSITION 3.8.

Let B be a Galois algebra over R with Galois group G and C the center of B. Then B satisfies the fundamental theorem if and only if B is one of the following three types: (1) B is an indecomposable commutative Galois algebra, (2) $B = Re \oplus R(1-e)$ where e and $1-e$ are minimal central idempotents in B, and (3) B is an indecomposable Galois algebra such that for each separable subalgebra A, $V_B(A) = \oplus \sum_{g \in G(A)} J_g$, and the centers of A and $B^{G(A)}$ are the same.

Now we derive a classification for Azumaya Galois extensions satisfying the fundamental theorem.

THEOREM 3.9.

Let B be an Azumaya Galois extension of B^G with Galois group G. Then B satisfies the fundamental theorem if and only if B is one of the following three types: (1) B is an indecomposable DeMeyer-Kanzaki Galois extension, (2) $B = B^G e \oplus B^G(1-e)$ where e and $1-e$ are minimal central idempotents in B, and (3) B is an indecomposable such that for each separable extension A of B^G in B, $V_B(A) = \oplus \sum_{g \in G(A)} J_g$, and the centers of A and $B^{G(A)}$ are the same.

PROOF. ($\Longrightarrow$) For convenience we denote $V_B(B^G)$ by Δ. Assume that B satisfies the fundamental theorem. Then by Theorem 3.7, Δ satisfies the fundamental theorem. Hence by Proposition 3.8, Δ is one of the following three types: (1) Δ is an indecomposable commutative Galois algebra, (2) $\Delta = C^G e \oplus C^G(1-e)$ where e and $1-e$ are minimal central idempotents in Δ, and (3) Δ is an indecomposable Galois algebra such that for each separable subalgebra S of Δ, $V_\Delta(S) = \oplus\sum_{g\in G(S)} J_g^\Delta$ where $J_g^\Delta = \{b \in \Delta \,|\, bx = g(x)b$ for each $x \in \Delta\}$, and the centers of S and $\Delta^{G(S)}$ are the same. In case (1) B is a center Galois extension with Galois group G by Theorem 3.5 in [15], and so B is a DeMeyer-Kanzaki Galois extension with Galois group G. Moreover, by Lemma 3.2, Δ and B have the same center C. Hence B is indecomposable. Therefore B is an indecomposable DeMeyer-Kanzaki Galois extension. In case (2), noting that $B = B^G \cdot \Delta \cong B^G \otimes_{C^G} \Delta$, we have that $B = B^G \cdot \Delta = B^G e \oplus B^G(1-e)$. But Δ and B have the same center C, so e and $1-e$ are minimal central idempotents in B. In case (3), since Δ and B have the same center C, B is indecomposable. Also, since B satisfies the fundamental theorem, for each separable extension A of B^G in B we have that $A = B^{G(A)}$, and so the centers of A and $B^{G(A)}$ are the same. Moreover, since B is a Galois extension of $B^{G(A)}$ with Galois group $G(A)$, $V_B(B^{G(A)}) = \oplus\sum_{g\in G(A)} J_g$ ([7], Proposition 1). Thus $V_B(A) = V_B(B^{G(A)}) = \oplus\sum_{g\in G(A)} J_g$.

($\Longleftarrow$) In case B is an indecomposable DeMeyer-Kanzaki Galois extension, B satisfies the fundamental theorem by Theorem 3 in [4]. In case $B = B^G e \oplus B^G(1-e)$ where e and $1-e$ are minimal central idempotents in B, noting that $B = B^G \cdot \Delta \cong B^G \otimes_{C^G} \Delta$, we have that $\Delta = C^G e \oplus C^G(1-e)$ where e and $1-e$ are minimal central idempotents in Δ. Hence Δ satisfies the fundamental theorem by Proposition 3.8. Thus B satisfies the fundamental theorem by Theorem 3.7. In case B is indecomposable such that for any separable extension A of B^G in B, $V_B(A) = \oplus\sum_{g\in G(A)} J_g$ and the centers of A and $B^{G(A)}$ are the same, we shall show that Δ is an indecomposable Galois algebra over C^G such that for each separable subalgebra S of Δ, $V_\Delta(S) = \oplus\sum_{g\in G(S)} J_g^\Delta$, and the centers of S and $\Delta^{G(S)}$ are the same; and so Δ satisfies the fundamental theorem by Proposition 3.8. In fact, since B and Δ have the same center C by Lemma 3.2, Δ is an indecomposable Galois algebra over C^G. Let S be a separable subalgebra of Δ. Then by Theorem 3.6, $B^G \cdot S$ ($\cong B^G \otimes_{C^G} S$) is a separable extension of B^G. Hence, by hypothesis, $V_B(B^G \cdot S) = \oplus\sum_{g\in G(B^G\cdot S)} J_g$, and $B^G \cdot S$ and $B^{G(B^G\cdot S)}$ have the same center. Clearly, $G(S) = G(B^G\cdot S)$ and $V_B(B^G\cdot S) = V_\Delta(S)$. Hence

$V_\Delta(S) = V_B(B^G \cdot S) = \oplus \sum_{g \in G(B^G \cdot S)} J_g = \oplus \sum_{g \in G(S)} J_g$. But $J_g = J_g^\Delta$ for each $g \in G$ ([14], Lemma 3.2), so $V_\Delta(S) = \oplus \sum_{g \in G(S)} J_g^\Delta$. Moreover, since $B^{G(B^G \cdot S)} = B^{G(S)}$ which is a separable extension of B^G by Lemma 3.4-(2), $B^{G(S)} = B^G \cdot V_{B^{G(S)}}(B^G) \cong B^G \otimes_{C^G} V_{B^{G(S)}}(B^G)$ by Lemma 3.4-(1) and Lemma 3.3. But $V_{B^{G(S)}}(B^G) = (V_B(B^G))^{G(S)} = \Delta^{G(S)}$, so $B^{G(B^G \cdot S)} = B^{G(S)} = B^G \cdot \Delta^{G(S)} \cong B^G \otimes_{C^G} \Delta^{G(S)}$. Now, since $B^G \cdot S$ ($\cong B^G \otimes_{C^G} S$) and $B^{G(B^G \cdot S)}$ ($\cong B^G \otimes_{C^G} \Delta^{G(S)}$) have the same center, S and $\Delta^{G(S)}$ have the same center. Hence by Proposition 3.8, Δ satisfies the fundamental theorem. Thus B satisfies the fundamental theorem by Theorem 3.7. This completes the proof.

Next, we shall give more correspondences for a commutator Galois extension. The correspondence given by Sugano ([10], Theorem 1) will play an important role. For convenience, we state it in the following:

PROPOSITION 3.10.

Let A be a H-separable extension of D (i.e., Hirata separable). Then if A is left or right D-finitely generated projective, there exists a one-to-one correspondence $V : E \longrightarrow V_A(E)$ such that V^2 is an identity between the set of separable extensions E of D in A such that E is a direct summand of A as a E-bimodule and the set of $Z(A)$-separable subalgebras of $V_A(D)$ where $Z(A)$ is the center of A.

LEMMA 3.11.

Let B be a commutator Galois extension of B^G with Galois group G. Then

(1) $B * G$ is H-separable over B^G and left B^G-finitely generated projective.

(2) The center of $B * G$ is C^G.

PROOF. (1) Since B is a commutator Galois extension of B^G with Galois group G, B is a Galois extension of B^G. Hence B is left B^G-finitely generated projective. Therefore, by the transitivity of finitely generated and projective modules, $B * G$ is left B^G-finitely generated projective since $B * G$ is left B-finitely generated projective. Moreover, since B is a commutator Galois extension of B^G with Galois group G again, B is finitely generated projective as a B^G-bimodule by Lemma 3.1. Thus $\mathrm{Hom}_{B^G}(B, B)$ is a H-separable extension of B^G ([11], Theorem 6). But B is a Galois extension of B^G with Galois group G, so $B * G \cong \mathrm{Hom}_{B^G}(B, B)$ ([4], Theorem 1). Thus $B * G$ is a H-separable extension of B^G.

(2) Since B is a commutator Galois extension of B^G with Galois group G, by Lemma 3.2, $B \cong B^G \otimes_{C^G} V_B(B^G)$ by the multiplication map and $V_B(B^G)$ is finitely generated projective over C^G. But C^G is commutative with 1, so C^G is a direct summand of $V_B(B^G)$ as a C^G-bimodule. Thus B^G is a direct summand B as a B^G-bimodule. Moreover, since B is a direct summand of $B * G$ as a B^G-bimodule, we have that B^G is a direct summand of $B * G$ as a B^G-bimodule. Now by part (1), $B * G$ is a Hirata separable extension of B^G. Hence $V_{B*G}(V_{B*G}(B^G)) = B^G$ ([9], Theorem 1.2). This implies that $B * G$ and B^G have the same center. But the center of B^G is C^G by Lemma 3.2, so the center of $B * G$ is C^G.

THEOREM 3.12.

Let B be a commutator Galois extension of B^G with Galois group G. Then there exists a one-to-one correspondence between the set of separable subalgebras of the Azumaya C^G-algebra $V_B(B^G) * G$ and the set of separable extensions E of B^G in $B * G$ such that E is a direct summand of $B * G$ as a E-bimodule.

PROOF. Since B is a commutator Galois extension of B^G with Galois group G, $V_B(B^G)$ is a Galois algebra over C^G with Galois group $G|_{V_B(B^G)} \cong G$ ([14], Lemma 3.1). Hence $V_B(B^G) * G$ is an Azumaya algebra over C^G. Moreover, by Lemma 3.11, $B * G$ is H-separable over B^G and left B^G-finitely generated projective. Thus, by Proposition 3.10, there exists a one-to-one correspondence between the set of separable extensions E

of B^G in $B*G$ such that E is a direct summand of $B*G$ as a E-bimodule and the set of separable subalgebras of the Azumaya C^G-algebra $V_B(B^G)*G$ ($= V_{B*G}(B^G)$ by Lemma 3.2).

LEMMA 3.13.

Let B be a commutator Galois extension of B^G with Galois group G. Then $B*G$ is H-separable over $(B*G)^{\overline{G}}$ and left (or right) $(B*G)^{\overline{G}}$-finitely generated projective.

PROOF. Since B is a commutator Galois extension of B^G with Galois group G, B is a Galois extension of B^G with Galois group G. Hence $B*G$ is a Galois extension of $(B*G)^{\overline{G}}$ with Galois group $\overline{G}$ with the same Galois system for B; and so $B*G$ is left (or right) finitely generated projective over $(B*G)^{\overline{G}}$. Moreover, since the elements in $\overline{G}$ are inner, $B*G$ is H-separable over $(B*G)^{\overline{G}}$ by Corollary 3 in [12].

THEOREM 3.14.

Let B be a commutator Galois extension of B^G with Galois group G of order n invertible in B. Then there exists a one-to-one correspondence between the set of C^G-separable subalgebras of C^GG and separable extensions E of $(B*G)^{\overline{G}}$ in $B*G$ such that E is a direct summand of $B*G$ as a E-bimodule.

PROOF. Since B is a commutator Galois extension of B^G with Galois group G, $B*G$ has center C^G by Lemma 3.11. Moreover, by Lemma 3.13, $B*G$ is H-separable over $(B*G)^{\overline{G}}$ and left (or right) $(B*G)^{\overline{G}}$-finitely generated projective, so by Proposition 3.10, there exists a one-to-one correspondence between the set of separable extensions E of $(B*G)^{\overline{G}}$ in $B*G$ such that E is a direct summand of $B*G$ as a E-bimodule and the set of C^G-separable subalgebras of $V_{B*G}((B*G)^{\overline{G}})$. Since n is invertible in C^G, C^GG is separable over C^G; and so C^GG is a C^G-separable subalgebra of $V_{B*G}((B*G)^{\overline{G}})$. Thus, by Proposition 3.10, $V_{B*G}(V_{B*G}(C^GG)) = C^GG$. But $V_{B*G}((B*G)^{\overline{G}}) = V_{B*G}(V_{B*G}(C^GG))$, so $V_{B*G}((B*G)^{\overline{G}}) = C^GG$. This completes the proof.

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