A Lifting Property of Split Exact Sequences of Bimodules over a Separable Ring Extension

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Abstract

Let A/D be a separable ring extension. Then it is shown that an exact sequence of A-bimodules $N \longrightarrow M \longrightarrow 0$ splits if it splits as D-bimodules.

Let A be a separable R-algebra. Then it is well known that a left A-module M is projective if it is projective as a left R-module ([1], Proposition 2.3). This is equivalent to that an exact sequence of left A-modules $N \longrightarrow M \longrightarrow 0$ splits if it splits as left R-modules. The purpose of the present paper is to generalize the the above result from separable algebra A over R to any separable ring extension A of D, where D is not necessarily contained in the center of A, for sequences of left modules and bimodules; that is, for a separable ring extension A of D, we shall show that the exact sequence of either left modules or bimodules over A, $N \longrightarrow M \longrightarrow 0$ splits if it splits either as left modules or bimodules over A.

Theorem 1.

Let A be a separable extension of D. Then an exact sequence of A-bimodules $N \longrightarrow M \longrightarrow 0$ splits if it splits as D-bimodules.

Proof. Let $\alpha : N \longrightarrow M$ be an onto A-bimodule homomorphism. By hypothesis, $\alpha : N \longrightarrow M$ as an onto D-bimodule homomorphism has a splitting D-bimodule homomorphism $\beta_0 : M \longrightarrow N$ such that $\alpha\beta_0 = 1_M$. We shall use β_0 to construct an A-bimodule homomorphism $\beta : M \longrightarrow N$ such that $\alpha\beta = 1_M$. Let Ω be the set of abelian group homomorphisms from M to N and define an additive map

$$\Phi : A \times A \longrightarrow \Omega$$

by $\Phi(a, a')(m) = a\beta_0(a'm)$ for all $a \times a' \in A \times A$ and $m \in M$. Then we show that Φ induces a map

$$\widetilde{\Phi} : A \otimes_D A \longrightarrow \Omega$$

by showing that Φ is bilinear and D-middle associative. In fact, it is easy to see that

$$\Phi(a + b, a') = \Phi(a, a') + \Phi(b, a') \text{ for all } a, b, a' \in A \text{ and}$$

$$\Phi(a, a' + b') = \Phi(a, a') + \Phi(a, b') \text{ for all } a, a', b' \in A.$$

Moreover, for any $d \in D$, and $a, a' \in A$, $d\beta_0(a'm) = \beta_0(da'm)$ since β_0 is a D-bimodule homomorphism from M to N. Thus

$$\Phi(ad, a')(m) = ad\beta_0(a'm) = a\beta_0(da'm) = \Phi(a, da')(m) \text{ for all } m \in M.$$

Therefore $\Phi(ad, a') = \Phi(a, da')$ for any $d \in D$ and $a, a' \in A$. This implies that Φ induces a map

$$\widetilde{\Phi} : A \otimes_D A \longrightarrow \Omega$$

where $\widetilde{\Phi}(a \otimes a') = \Phi(a, a')$ for $a \otimes a' \in A \otimes_D A$.

Next, for any $g \in \widetilde{\Phi}(A \otimes_D A)$, we define an additive map

$$\Psi_g : A \times A \longrightarrow \Omega.$$

by $\Psi_g(b, b')(m) = g(mb)b'$ for all $a \times a' \in A \times A$ and $m \in M$, and show that Ψ_g induces a map

$$\widetilde{\Psi}_g : A \otimes_D A \longrightarrow \Omega$$

by showing that Ψ is bilinear and D-middle associative. In fact, it is easy to see that

$$\Psi_g(a+b,a') = \Psi_g(a,a') + \Psi_g(b,a') \text{ for all } a,b,a' \in A \text{ and}$$

$$\Psi_g(a,a'+b') = \Psi_g(a,a') + \Psi_g(a,b') \text{ for all } a,a',b' \in A.$$

Moreover, since $g \in \widetilde{\Phi}(A \otimes_D A)$, there exist $\{a_i, a_i' \in A,\ i = 1,2,...,t$ for some integer $t\}$ such that $g = \widetilde{\Phi}(\sum_{i=1}^t a_i \otimes a_i')$ and $g(m) = \sum_{i=1}^t a_i\beta_0(a_i'm)$. Thus, for any $d \in D$,

$$\begin{aligned}
\Psi_g(bd,b')(m) &= g(m(bd))b' = \sum_{i=1}^t a_i\beta_0(a_i'mbd)b' \\
&= \sum_{i=1}^t a_i\beta_0(a_i'mb)db' \text{ (because } \beta_0 \text{ is a } D\text{-bimodule homomorphism)} \\
&= g(mb)db' \\
&= \Psi_g(b,db')(m)
\end{aligned}$$

for all $m \in M$. Therefore $\Psi_g(bd,b') = \Psi_g(b,db')$ for all $d \in D$ and $b,b' \in A$. This implies that Ψ_g induces a map

$$\widetilde{\Psi}_g : A \otimes_D A \longrightarrow \Omega$$

where $\widetilde{\Psi}_g(b \otimes b') = \Psi_g(b,b')$ for $b \otimes b' \in A \otimes_D A$. Now, since A is a separable extension of D, there exists a separable element $e = \sum_{i=1}^k a_i \otimes b_i \in A \otimes_D A$ such that $\sum a_ib_i = 1$, and $\sum aa_i \otimes b_i = \sum a_i \otimes b_ia$ for all a in A. Let $f = \widetilde{\Phi}(e)$ and $\beta = \widetilde{\Psi}_f(e)$. Then we claim that β is the splitting A-bimodule homomorphism of α. In fact, for any $m \in M$

$$\begin{aligned}
\alpha\beta(m) &= \alpha\widetilde{\Psi}_f(e)(m) = \alpha(\sum_{j=1}^k f(ma_j)b_j) \\
&= \alpha(\sum_{j=1}^k \widetilde{\Phi}(e)(ma_j)b_j) = \alpha(\sum_{j=1}^k\sum_{i=1}^k a_i\beta_0(b_ima_j)b_j) \\
&= \sum_{j=1}^k\sum_{i=1}^k a_i(\alpha\beta_0)(b_ima_j)b_j \text{ (because } \alpha \text{ is an } A\text{-bimodule homomorphism)} \\
&= \sum_{j=1}^k\sum_{i=1}^k a_i(b_ima_j)b_j \text{ (because } \alpha\beta_0 = 1_M) \\
&= (\sum_{i=1}^k a_ib_i)m(\sum_{j=1}^k a_jb_j) = m.
\end{aligned}$$

Thus we have $\alpha\beta = 1_M$. Moreover, since $\sum aa_i \otimes b_i = \sum a_i \otimes b_i a$ for all a in A, we have that $\widetilde{\Phi}(\sum aa_i \otimes b_i) = \widetilde{\Phi}(\sum a_i \otimes b_i a)$ and $\widetilde{\Psi}_f(\sum aa_i \otimes b_i) = \widetilde{\Psi}_f(\sum a_i \otimes b_i a)$. Hence for any $a \in A$ and $m \in M$,

$$\begin{aligned}
\beta(am) &= \widetilde{\Psi}_f(e)(am) = \sum_{j=1}^{k} f(ama_j)b_j \\
&= \sum_{j=1}^{k} \widetilde{\Phi}(e)(ama_j)b_j = \sum_{j=1}^{k}\sum_{i=1}^{k} a_i\beta_0(b_iama_j)b_j \\
&= \sum_{j=1}^{k} \widetilde{\Phi}(\sum a_i \otimes b_i a)(ma_j)b_j = \sum_{j=1}^{k} \widetilde{\Phi}(\sum aa_i \otimes b_i)(ma_j)b_j \\
&= \sum_{j=1}^{k}\sum_{i=1}^{k} aa_i\beta_0(b_ima_j)b_j = a\sum_{j=1}^{k}\sum_{i=1}^{k} a_i\beta_0(b_ima_j)b_j \\
&= a\sum_{j=1}^{k} \widetilde{\Phi}(e)(ma_j)b_j \\
&= a\sum_{j=1}^{k} f(ma_j)b_j = a\widetilde{\Psi}_f(e)(m) \\
&= a\beta(m).
\end{aligned}$$

Also,

$$\begin{aligned}
\beta(ma) &= \widetilde{\Psi}_f(e)(ma) = \sum_{j=1}^{k} f(maa_j)b_j \\
&= \sum_{j=1}^{k} f(m(aa_j))b_j = \widetilde{\Psi}_f(\sum aa_i \otimes b_i)(m) \\
&= \widetilde{\Psi}_f(\sum a_i \otimes b_i a)(m) = \sum_{j=1}^{k} f(ma_j)b_j a \\
&= \widetilde{\Psi}_f(e)(m)a = \beta(m)a.
\end{aligned}$$

Therefore $\beta : M \longrightarrow N$ is an A-bimodule homomorphism such that $\alpha\beta = 1_M$.

By the proof of Theorem 1, we have the following corollary.

Corollary 1.

Let A be a separable extension of D. Then an exact sequence of left A-modules $N \longrightarrow M \longrightarrow 0$ splits if it splits as left D-modules.

Corollary 2.

Let A be a separable extension of D. Then a left A-module M is projective if it is projective as a left D-module.

Proof. Since any left module M over a ring B is a homomorphism image of a direct sum of B's, that M is projective over B is equivalent to that every exact left B-modules sequence $N \longrightarrow M \longrightarrow 0$ splits. Thus Corollary 2 is immediate by Corollary 1.

Remark 1.

Corollary 2 is a generalization of the lifting property of projective modules over a separable extension from a separable algebra ([1], Proposition 2.3).

Remark 2.

Professor Shuichi Ikehata provides a different proof of Corollary 2 as following: Since ${}_DM$ is projective, there exists a dual bases for ${}_DM$, $\{m_i \in M,\, g_i \in \mathrm{Hom}({}_DM, {}_DD) \,|\, i \in I$ for some index set $I\}$ such that $\sum_i g_i(m)m_i = m$ for all $m \in M$. Since A is a separable extension of D, there exist $\{x_j, y_j$ in A; $j = 1, 2, ..., k$ for some integer $k\}$ such that $\sum x_j y_j = 1$, and $\sum a x_j \otimes y_j = \sum x_j \otimes y_j a$ for all a in A where $\otimes$ is over D. Noting that $\Psi : A \longrightarrow A \otimes_D A$ by $\Psi(a) = \sum a x_j \otimes y_j = \sum x_j \otimes y_j a$ is an A-bimodule homomorphism, we define $\widetilde{g}_i : {}_AM \longrightarrow {}_AA$ as a composition of the following homomorphisms,

$$ {}_AM \longrightarrow {}_AA \otimes_A M \xrightarrow{\Psi \otimes 1_M} {}_AA \otimes_D A \otimes_A M \longrightarrow {}_AA \otimes_D M \xrightarrow{1_A \otimes g_i} {}_AA \otimes_D D \longrightarrow {}_AA \quad \text{by} $$

$$ m \longmapsto 1 \otimes m \longmapsto \sum_j x_j \otimes y_j \otimes m \longmapsto \sum_j x_j \otimes (y_j m) \longmapsto \sum_j x_j \otimes g_i(y_j m) \longmapsto \sum_j x_j g_i(y_j m), $$

that is, $\widetilde{g}_i(m) = \sum_j x_j g_i(y_j m)$ for each $m \in M$. We can see that for each $m \in M$

$$\sum_i \widetilde{g}_i(m) m_i = \sum_i \sum_j x_j g_i(y_j m) m_i = \sum_j x_j (\sum_i g_i(y_j m) m_i) = \sum_j x_j (y_j m) = m.$$

Thus $\{m_i, \widetilde{g}_i\}$ is a dual bases for ${}_A M$, that is, ${}_A M$ is projective.

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References

[1] F.R. DeMeyer and E. Ingraham, Separable Algebras Over Commutative Rings, Vol. 181, Springer-Verlag, Berlin,1971.