# ON CHARACTERIZATIONS OF A COMMUTATOR GALOIS EXTENSION

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**Abstract**

Let $B$ be a ring with 1, $C$ the center of $B$, $G$ a finite automorphism group of $B$, $B^G$ the set of elements in $B$ fixed under each element in $G$, and $V_B(B^G)$ the commutator subring of $B^G$ in $B$. Then $B$ is called a commutator Galois extension with Galois group $G$ if $V_B(B^G)$ is a Galois extension with Galois group $G|_{V_B(B^G)} \cong G$. It is shown that $B$ is a commutator Galois extension with Galois group $G$ if and only if $B$ is a Galois extension with Galois group $G$ such that $B = B^G \cdot V_B(B^G)$. This generalizes a result for center Galois extensions.

## 1. Introduction

Let $B$ be a Galois extension of $B^G$ with Galois group $G$ and $C$ the center of $B$. In [4], F. DeMeyer showed that, if $C$ is a Galois extension with Galois group $G|_C \cong G$ (that is, $B$ is a center Galois extension with Galois group $G$), then $B = B^G C$ ([4], Lemma 2). In [9], it was shown that $B$ is a center Galois extension with Galois group $G$ if and only if $B$ is a Galois extension with Galois group $G$ such that $B = B^G C$ ([9], Theorem 3.2). Noting that $C \subset V_B(B^G)$, the commutator subring of $B^G$ in $B$, we have that a center Galois

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extension $B$ with Galois group $G$ is a commutator Galois extension with Galois group $G$ (that is, $V_B(B^G)$ is a Galois extension with Galois group $G|_{V_B(B^G)} \cong G$). The purpose of the present paper is to generalize the above result from a center Galois extension to a commutator Galois extension. We shall show that $B$ is a commutator Galois extension with Galois group $G$ is equivalent to each of the following statements:

(1) $B$ is a Galois extension with Galois group $G$ such that $B = B^G \cdot V_B(B^G)$.

(2) The skew group ring $B*G$ is a Hirata separable extension of $B^G$ and $B = B^G \cdot V_B(B^G)$ which contains $B^G$ as a direct summand as a $B^G$-bimodule.

Also, by applying a result for a Galois algebra as given in [11], we have a structure for a commutator Galois extension $B$ of $B^G$ with Galois group $G$ such that $G = \text{Aut}_{B^G}(B)$; that is, either $B$ is a center Galois extension with no central idempotents but 0 and 1, or $B = B^G \oplus B^G$ where $B^G$ contains no central idempotents but 0 and 1.

## 2 Basic Definitions and Notations

Throughout this paper, $B$ will represent a ring with 1, $C$ the center of $B$, $G$ a finite automorphism group of $B$, $B^G$ the set of elements in $B$ fixed under each element in $G$, and $B * G$ the skew group ring of $G$ over $B$, that is, $B * G$ is the free left $B$-module in which the multiplication is given by $gb = g(b)g$ for $b \in B$ and $g \in G$.

Let $A$ be a subring of a ring $B$ with the same identity 1. We denote $V_B(A)$ the commutator subring of $A$ in $B$. We call $B$ a separable extension of $A$ if there exist $\{a_i, b_i$ in $B$, $i = 1, 2, ..., k$ for some integer $k\}$ such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all $b$ in $B$ where $\otimes$ is over $A$. An Azumaya algebra is a separable extension of its center. A ring $B$ is called a Hirata separable extension of $A$ if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of $B$ as a $B$-bimodule. We call $B$ a Galois extension of $B^G$ with Galois group $G$ if there exist elements $\{a_i, b_i$ in $B$, $i = 1, 2, ..., m\}$ for some integer $m$ such that $\sum_{i=1}^{m} a_i g(b_i) = \delta_{1,g}$ for each $g \in G$ ([4]). Such a set $\{a_i, b_i\}$ is called a $G$-Galois system for $B$. A Galois extension $B$ of $B^G$ is called a Galois algebra over $B^G$ if $B^G$ is contained in $C$ ([10]). We called $B$ a center Galois extension with Galois group $G$ if $C$ is a Galois algebra over $C^G$ with Galois group $G|_C \cong G$ ([6], [9]), and a commutator Galois extension of $B^G$ with Galois group $G$ if $V_B(B^G)$ is a Galois extension of $(V_B(B^G))^G$ with Galois group $G|_{V_B(B^G)} \cong G$. A Galois extension $B$ of $B^G$ with Galois group $G$ is called an Azumaya Galois extension if $B^G$ is an Azumaya $C^G$-algebra ([1], [2]).

## 3. Main Results

Keeping the definitions and notations in section 2, in this section, we shall give two characterizations of a commutator Galois extension. We begin with some properties of a Galois extension $B$ such that $B = B^G \cdot V_B(B^G)$.

**Lemma 3.1.** *Let $B$ be a Galois extension of $B^G$ with Galois group $G$ such that $B = B^G \cdot V_B(B^G)$. Then*

*(1) $B$ has a Galois system $\{b_i \in B; d_i \in V_B(B^G), i = 1, 2, ..., n\}$ for some integer $n$.*

*(2) $B$ is finitely generated projective as a $B^G$-bimodule.*

*(3) The center of $V_B(B^G)$ is $C$ and the center of $B^G$ is $C^G$.*

*(4) $B \cong B^G \otimes_{C^G} V_B(B^G)$ by the multiplication map and $V_B(B^G)$ is finitely generated projective over $C^G$.*

**Proof.** (1) Since $B$ is a Galois extension of $B^G$ with Galois group $G$, $B$ has a Galois system $\{x_i \in B; y_i \in B, i = 1, 2, ..., m\}$ for some integer $m$, that is, $\sum_{i=1}^{m} x_i g(y_i) = \delta_{1,g}$ for each $g \in G$. By hypothesis, $B = B^G \cdot V_B(B^G)$, so $y_i = \sum_{k=1}^{n_i} a_k^{(i)} b_k^{(i)}$ for some $a_k^{(i)} \in B^G$ and $b_k^{(i)} \in V_B(B^G)$, $k = 1, 2, ..., n_i$, $i = 1, 2, ..., m$. Therefore,

$$\delta_{1,g} = \sum_{i=1}^{m} x_i g(y_i) = \sum_{i=1}^{m} x_i g(\sum_{k=1}^{n_i} a_k^{(i)} b_k^{(i)}) = \sum_{i=1}^{m}\sum_{k=1}^{n_i} x_i g(a_k^{(i)} b_k^{(i)}) = \sum_{i=1}^{m}\sum_{k=1}^{n_i} x_i a_k^{(i)} g(b_k^{(i)}).$$

Let $b_{ik} = x_i a_k^{(i)}$ and $d_{ik} = b_k^{(i)}$, $k = 1, 2, ..., n_i$, $i = 1, 2, ..., m$. Then $\sum_{i=1}^{m}\sum_{k=1}^{n_i} b_{ik} g(d_{ik}) = \delta_{1,g}$ for each $g \in G$. This shows that

$$\{b_{ik} \in B; d_{ik} \in V_B(B^G), k = 1, 2, ..., n_i, i = 1, 2, ..., m\}$$

is a Galois system for $B$.

(2) By (1), $B$ has a Galois system $\{b_i \in B;\ d_i \in V_B(B^G),\ i = 1, 2, ..., n\}$ for some integer $n$, that is, $\sum_{i=1}^{n} b_i g(d_i) = \delta_{1,g}$ for each $g \in G$. Let $f_i : B \longrightarrow B^G$ given by $f_i(b) = \sum_{g\in G} g(d_i b)$ for all $b \in B$, $i = 1, 2, \cdots n$. Then, it is easy to check that $f_i$ is a homomorphism as $B^G$-bimodule. Moreover, for any $b \in B$,

$$\sum_{i=1}^{m} b_i f_i(b) = \sum_{i=1}^{m} b_i \sum_{g\in G} g(d_i b) = \sum_{g\in G}\sum_{i=1}^{m} b_i g(d_i b) = \sum_{g\in G} \big(\sum_{i=1}^{m} b_i g(d_i)\big) g(b) = b.$$

Hence $\{b_i;\ f_i,\ i = 1, 2, ..., m\}$ is a dual bases for $B$ as $B^G$-bimodule, and so $B$ is finitely generated projective as a $B^G$-bimodule.

(3) Let $Z$ be the center of $V_B(B^G)$. Then clearly, $C \subset Z$. By hypothesis, $B = B^G \cdot V_B(B^G)$. Hence for any $z \in Z$, we have $z \in C$, that is, $Z \subset C$. Thus $Z = C$. Similarly, since $B = B^G \cdot V_B(B^G)$, the center of $B^G$ is contained in $C$. Therefore the center of $B^G$ is $C^G$.

(4) By part (2), $B$ is finitely generated projective as a $B^G$-bimodule. Noting that the center of $B^G$ is $C^G$ by part (3), we have that $B \cong B^G \otimes_{C^G} V_B(B^G)$ as $B^G$-bimodules by the multiplication map $a \otimes b \longrightarrow ab$ for $a \otimes b \in B^G \otimes_{C^G} V_B(B^G)$ and $V_B(B^G)$ is finitely generated projective over $C^G$ ([5], Proposition 5.2). It is easy to check that the isomorphism $B \cong B^G \otimes_{C^G} V_B(B^G)$ by the multiplication map is also a ring isomorphism.

**Lemma 3.2.** *Let $B$ be a commutator Galois extension of $B^G$ with Galois group $G$. Then*

*(1) $B$ is a Galois extension of $B^G$ with Galois group $G$ such that $B = B^G \cdot V_B(B^G)$.*

*(2) $V_B(B^G)$ is a Galois algebra over $C^G$ with Galois group $G|_{V_B(B^G)} \cong G$.*

**Proof.** (1) Since $B$ is a commutator Galois extension of $B^G$ with Galois group $G$, $V_B(B^G)$ is a Galois extension of $(V_B(B^G))^G$ with Galois group $G|_{V_B(B^G)} \cong G$. Noting that $V_B(B^G) \subset B^G \cdot V_B(B^G) \subset B$, we have that both $B^G \cdot V_B(B^G)$ and $B$ are Galois extensions of $B^G$ with Galois group $G|_{B^G \cdot V_B(B^G)} \cong G$. Thus $B = B^G \cdot V_B(B^G)$.

(2) Since $B$ is a commutator Galois extension with Galois group $G$, $V_B(B^G)$ is a Galois extension of $(V_B(B^G))^G$. By part (1), $B$ is a Galois extension of $B^G$ with Galois group $G$ such that $B = B^G \cdot V_B(B^G)$, so the center of $B^G$ is $C^G$ by Lemma 3.1-(3). Hence $(V_B(B^G))^G = V_{B^G}(B^G) = C^G$. Therefore $V_B(B^G)$ is a Galois algebra over $C^G$ with Galois group $G|_{V_B(B^G)} \cong G$.

Now we show the main theorem.

**Theorem 3.3.** The following statements are equivalent:

*(1) $B$ is a commutator Galois extension with Galois group $G$.*

*(2) $B$ is a Galois extension of $B^G$ with Galois group $G$ such that $B = B^G \cdot V_B(B^G)$.*

*(3) $B * G$ is a Hirata separable extension of $B^G$ and $B = B^G \cdot V_B(B^G)$ which contains $B^G$ as a direct summand as a $B^G$-bimodule.*

**Proof.** (1) $\Longrightarrow$ (2) Since $B$ is a commutator Galois extension with Galois group $G$, $B$ is a Galois extension with Galois group $G$ such that $B = B^G \cdot V_B(B^G)$ by Lemma 3.2-(1).

(2) $\Longrightarrow$ (3) Since $B$ is a Galois extension with Galois group $G$ such that $B = B^G \cdot V_B(B^G)$, $B$ is finitely generated projective as a $B^G$-bimodule by Lemma 3.1-(2). Thus $\mathrm{Hom}_{B^G}(B,B)$ is a Hirata separable extension of $B^G$ ([8], Theorem 6). But $B$ is a Galois extension of $B^G$ with Galois group $G$, so $B * G \cong \mathrm{Hom}_{B^G}(B,B)$ ([4], Theorem 1). Thus $B * G$ is a Hirata separable extension of $B^G$. Next we claim that $B = B^G \cdot V_B(B^G)$ which contains $B^G$ as a direct summand as a $B^G$-bimodule. In fact, since $B$ is a Galois extension with Galois group $G$ such that $B = B^G \cdot V_B(B^G)$, $B \cong B^G \otimes_{C^G} V_B(B^G)$ by the multiplication map and $V_B(B^G)$ is finitely generated projective over $C^G$ by Lemma 3.1-(4). But $C^G$ is commutative with 1, so $V_B(B^G)$ is a progenerator of $C^G$. Hence $C^G$ is a direct summand of $V_B(B^G)$ as a $C^G$-bimodule ([3], Corollary 4.2, page 56). Therefore $B$ ($\cong B^G \otimes_{C^G} V_B(B^G)$) contains $B^G$ as a direct summand as a $B^G$-bimodule. This completes the proof.

(3) $\Longrightarrow$ (1) By hypothesis, $B^G$ is a direct summand $B$ as a $B^G$-bimodule. Noting that $B$ is a direct summand of $B*G$ as a $B^G$-bimodule, we have that $B^G$ is a direct summand of $B*G$ as a $B^G$-bimodule. Moreover, $B*G$ is a Hirata separable extension of $B^G$ by hypothesis. Hence $V_{B*G}(B^G)$ is separable over the center of $B*G$ and $V_{B*G}(V_{B*G}(B^G)) = B^G$ ([7], Proposition 1.3-(1)). Clearly, the center of $B*G$ is contained in the center of $V_{B*G}(B^G)$. Thus $V_B(B^G)*G$ $(= V_{B*G}(B^G))$ is an Azumaya algebra ([3], Theorem 3.8, page 55). Also, $V_{B*G}(V_{B*G}(B^G)) = B^G$ implies that $V_{B*G}(B^G)$ and $B^G$ have the same center. But the center of $B^G$ is contained in $V_B(B^G)$, so the center of $V_B(B^G)*G$ $(= V_{B*G}(B^G))$ is contained in $V_B(B^G)$. Hence $V_B(B^G)$ is an Azumaya Galois extension with Galois group $G|_{V_B(B^G)} \cong G$ ([2], theorem 3.1); and so $B$ is a commutator Galois extension with Galois group $G$.

Let $B$ be a commutator Galois extension of $B^G$ with Galois group $G$. Then $V_B(B^G)$ is a Galois algebra over $C^G$ with Galois group $G|_{V_B(B^G)} \cong G$ by Lemma 3.2-(2). Hence the result for a Galois algebra as given in [11] can be applied to a commutator Galois extension $B$ such that $G = \mathrm{Aut}_{B^G}(B)$. We first give a lemma.

**Lemma 3.4.** *Let $B$ be a commutator Galois extension with Galois group $G$. Then $B$ is a center Galois extension of $B^G$ with Galois group $G$ if and only if $V_B(B^G) = C$.*

**Proof.** See Theorem 3.5 in [12].

**Theorem 3.5.** *Let $B$ be a commutator Galois extension of $B^G$ with Galois group $G$. Then $G = Aut_{B^G}(B)$ if and only if either $B$ is a center Galois extension with no central idempotents but 0 and 1, or $B = B^G \oplus B^G$ where $B^G$ contains no central idempotents but 0 and 1.*

**Proof.** Since $B$ is a commutator Galois extension of $B^G$ with Galois group $G$, $V_B(B^G)$ is a Galois algebra over $C^G$ with Galois group $G|_{V_B(B^G)} \cong G$ by Lemma 3.2-(2). Hence $G|_{V_B(B^G)} = \mathrm{Aut}_{C^G}(V_B(B^G))$ if and only if either $V_B(B^G)$ is a commutative Galois algebra with no idempotents but 0 and 1, or $V_B(B^G) = C^G \oplus C^G$ where $C^G$ contains no idempotents but 0 and 1 ([11], Theorem 4.6). Since $B$ is a commutator Galois extension of $B^G$ with Galois group $G$ again, $B = B^G \cdot V_B(B^G) \cong B^G \otimes_{C^G} V_B(B^G)$ by the multiplication map by Lemma 3.1-(4). Hence $\mathrm{Aut}_{B^G}(B) \cong \mathrm{Aut}_{C^G}(V_B(B^G))$ by the map $\sigma \longrightarrow \sigma|_{V_B(B^G)}$ for each $\sigma \in \mathrm{Aut}_{B^G}(B)$. Noting that the center of $V_B(B^G)$ is $C$ by Lemma 3.1-(3), we have that $V_B(B^G)$ is commutative if and only if $V_B(B^G) = C$. Hence by Lemma 3.4, $V_B(B^G)$ is commutative if and only if $B$ is a center Galois extension of $B^G$ with Galois group $G$. Also, by Lemma 3.1-(3), the center of $B^G$ is $C^G$. Thus $G = \mathrm{Aut}_{B^G}(B)$ if and only if either $B$ is a center Galois extension with no central idempotents but 0 and 1, or $B = B^G \oplus B^G$ where $B^G$ contains no central idempotents but 0 and 1.

We recall that an Azumaya Galois extension $B$ is a Galois extension of $B^G$ such that $B^G$ is an Azumaya $C^G$-algebra. In [1], it was shown that, if $B$ is an Azumaya Galois extension with Galois group $G$, then $B$ is a commutator Galois extension with Galois group $G$ ([1], Theorem 2). Hence the class of commutator Galois extensions is a broader class than the class of Azumaya Galois extensions. Next we give some conditions under which a commutator Galois extension becomes an Azumaya Galois extension.

**Theorem 3.6.** *Let $B$ be a commutator Galois extension of $B^G$ with Galois group $G$. Then the following statements are equivalent:*
*(1) $B$ is an Azumaya Galois extension of $B^G$ with Galois group $G$.*
*(2) $B^G$ is separable over $C^G$.*
*(3) $B$ is separable over $C^G$.*
*(4) $B$ is an Azumaya algebra over $C$.*

**Proof.** (1) $\Longrightarrow$ (2) Since $B$ is an Azumaya Galois extension of $B^G$ with Galois group $G$, $B^G$ is an Azumaya $C^G$-algebra, and so $B^G$ is separable over $C^G$.

(2) $\Longrightarrow$ (3) Since $B$ is a Galois extension of $B^G$ with Galois group $G$, $B$ is separable

over $B^G$. But $B^G$ is separable over $C^G$ by hypothesis, so $B$ is separable over $C^G$ by the transitivity property of separable extensions.

(3) $\Longrightarrow$ (4) $B$ is separable over $C^G$, $B$ is an Azumaya algebra over $C$ ([3], Theorem 3.8, page 55).

(4) $\Longrightarrow$ (1) By hypothesis, $B$ is a commutator Galois extension of $B^G$ with Galois group $G$. Moreover, $B$ is an Azumaya algebra over $C$, so $B^G$ is an Azumaya $C^G$-algebra ([12], Theorem 3.1). Thus $B$ is an Azumaya Galois extension of $B^G$ with Galois group $G$.

We conclude this paper with an example to illustrate that a commutator Galois extension is not necessarily a center Galois extension or an Azumaya Galois extension.

**Example.** Let $Q[i,j,k]$ be the quaternion algebra over the rational field $Q$, $A$ a noncommutative ring with center $Q$ but not separable over $Q$, $B = A \otimes_Q Q[i,j,k]$, and $G = \{1, g_i, g_j, g_k\}$ where $g_i(a \otimes x) = a \otimes ixi^{-1}$, $g_j(a \otimes x) = a \otimes jxj^{-1}$, and $g_k(a \otimes x) = a \otimes kxk^{-1}$ for all $a \in A$ and $x \in Q[i,j,k]$. Then

(1) It is easy to check that $B^G = A \otimes_Q Q \cong A$.

(2) $V_B(B^G) = Q \otimes_Q Q[i,j,k] \cong Q[i,j,k]$.

(3) $V_B(B^G)$ is a Galois extension with Galois group $G|_{V_B(B^G)} \cong G$ with a Galois system $\{1 \otimes 1, 1 \otimes i, 1 \otimes j, 1 \otimes k;\ \frac{1}{4} \otimes 1, -\frac{1}{4} \otimes i, -\frac{1}{4} \otimes j, -\frac{1}{4} \otimes k\}$, so $B$ is a commutator Galois extension of $B^G$ with Galois group $G$.

(4) Since $B^G = A \otimes_Q Q \cong A$ which is not separable over its center $Q$, $B^G$ is not an Azumaya algebra; and so $B$ is not an Azumaya Galois extension with Galois group $G$.

(5) The center $C$ of $B$ is $Q \otimes_Q Q \cong Q$ which is not a Galois extension with Galois group $G|_C \cong G$. Hence $B^G$ is not a center Galois extension with Galois group $G$.

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