On Splitting Rings for Azumaya Skew Group Rings

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Abstract: Let B be a ring with 1, G an automorphism group of B of order n for some integer n, $B*G$ the skew group ring over B with a free basis $\{g|\ g \in G\}$, B^G the set of elements in B fixed under G, and $\bar{G}$ the inner automorphism group of $B*G$ induced by G. It is shown that when C is a G-Galois algebra over C^G with Galois group $G|_C \cong G$ or B is a G-Galois extension of B^G and $n^{-1} \in B$, then, $B*G$ is an Azumaya algebra if and only if so is $(B*G)^{\bar{G}}$, and some splitting rings of $B*G$, $(B*G)^{\bar{G}}$ and B are shown to be the same.

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1 – Introduction

Let B be a ring with 1, C the center of B, G an automorphism group of B of order n for some integer n, $B*G$ a skew group ring over B with a free basis $\{g \mid g \in G\}$, B^G the set of elements in B fixed under G, $\bar{G}$ the inner automorphism group of $B*G$ induced by G, that is, $\bar{g}(f) = gfg^{-1}$ for each $f \in B*G$ and $g \in G$. We note that $\bar{G}$ restricted to B is G.

In [1] and [2], the Azumaya skew group ring $B*G$ over C^G was characterized in terms of Azumaya Galois extension B of B^G and the H-separable extension $B*G$ of B respectively. Also in [3], the commutator subring of B in $B*G$ was studied. In the present paper, under a Galois condition on B, the Azumaya skew group ring $B*G$ is characterized in terms of the Azumaya fixed subring $(B*G)^{\bar{G}}$ under $\bar{G}$ and the Azumaya coefficient ring B, that is, when C is a G-Galois algebra over C^G with Galois group $G|_C \cong G$ or B is a G-Galois extension of B^G and $n^{-1} \in B$, then, $B*G$ is an Azumaya algebra if and only if so is $(B*G)^{\bar{G}}$.

Let A be an Azumaya algebra. It is well known that any separable maximal commutative subalgebra of A is a splitting ring for A ([4], Theorem 5.5, p. 64). In this paper, we call F a splitting ring for A if F is a separable maximal commutative subalgebra of A. We then show that when C is a G-Galois algebra over C^G with Galois group $G|_C \cong G$, F is a splitting ring for the Azumaya algebra $B * G$ containing C if and only if F is a splitting ring for the Azumaya algebra B. Moreover, when B is a G-Galois extension of B^G and $n^{-1} \in B$, F is a splitting ring for the Azumaya algebra $B * G$ containing the center of $(B * G)^{\bar{G}}$, then, F is a splitting ring for $(B * G)^{\bar{G}}$ if and only if G is Abelian. At the end, two examples are constructed to demonstrate the results. This paper was written under the support of a Caterpillar Fellowship at Bradley University. We would like to thank Caterpillar Inc. for the support.

2 – Basic Definitions and Notations

Throughout this paper, B will represent a ring with 1, G an automorphism group of B, C the center of B, $B * G$ a skew group ring in which the multiplication is given by $gb = g(b)g$ for $b \in B$ and $g \in G$, B^G the set of elements in B fixed under G, Z the center of $B * G$, $\bar{G}$ the inner automorphism group of $B * G$ induced by G, that is, $\bar{g}(f) = gfg^{-1}$ for each $f \in B * G$ and $g \in G$. We note that $\bar{G}$ restricted to B is G.

Let A be a subring of a ring B with the same identity 1. We denote $V_B(A)$ the commutator subring of A in B. We call B a separable extension of A if there exist $\{a_i, b_i$ in B, $i = 1, 2, ..., m$ for some integer $m\}$ such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where $\otimes$ is over A, and a ring B is called a H-separable extension of A if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B-bimodule. An Azumaya algebra is a separable extension of its center. B is called a G-Galois extension of B^G if there exist elements $\{c_i, d_i$ in B, $i = 1, 2, ..., m\}$ for some integer m such that $\sum_{i=1}^{m} c_i g(d_i) = \delta_{1,g}$. The set $\{c_i, d_i\}$ is called a G-Galois system for B. B is called a DeMeyer-Kanzaki G-Galois extension if B is an Azumaya C-algebra and C is G-Galois algebra with $G|_C \cong G$. If A is an Azumaya C-algebra and S is a commutative C-algebra such that $A \otimes_C S \cong \mathrm{Hom}_S(E, E)$ for some S-progenerator E, then S is called a splitting ring for the Azumaya algebra A. It is well known that any separable maximal commutative subalgebra of A is a splitting ring for A. In the present paper, S is called a splitting ring for A if S is a separable maximal commutative subalgebra of A.

3 – Characterizations of Azumaya Skew Group Rings

In this section, we shall characterize an Azumaya skew group ring $B * G$ in terms of $(B * G)^{\bar{G}}$ and B under a Galois condition that C is a G-Galois algebra over C^G with Galois group $G|_C \cong G$ or B is a G-Galois extension of B^G and $n^{-1} \in B$. We begin with a Lemma.

Lemma 3.1. If C is a G-Galois algebra over C^G with Galois group $G|_C \cong G$, then
(a) $B * G$ is H-separable over B.
(b) $B * G$ is H-separable over $(B * G)^{\bar{G}}$.
(c) The center of $B * G$, $Z = C^G$.
(d) $V_{B*G}(C) = B$.

Proof: (a) Since C is a G-Galois algebra over C^G with Galois group $G|_C \cong G$ and $C \subseteq V_{B*G}(B)$, $V_{B*G}(B)$ is $\bar{G}$-Galois extension of $(V_{B*G}(B))^{\bar{G}}$ with the same Galois system as C. Hence, $B * G$ is H-separable extension of B by ([3], Theorem 1).

(b) Since C is a G-Galois extension of C^G with Galois group $G|_C \cong G$, $B * G$ is a $\bar{G}$-Galois extension of $(B * G)^{\bar{G}}$ with the same Galois system as C. But $\bar{G}$ acts on $B * G$ is inner, so $B * G$ is H-separable extension of $(B * G)^{\bar{G}}$ by ([7], Corollary 3).

(c) By (a), $B * G$ is H-separable over B. Moreover, B is a direct summand of $B * G$ as a left B-module, so B satisfies the double centralizer property in $B * G$ ([8], Proposition 1.2), that is, $B = V_{B*G}(V_{B*G}(B))$. This implies that the center of $B * G$ is contained in B. Thus, $Z = C^G$.

(d) Clearly, $B \subseteq V_{B*G}(C)$. Conversely, for each $\sum_{g \in G} b_g g$ in $V_{B*G}(C)$, we have $c(\sum_{g \in G} b_g g) = (\sum_{g \in G} b_g g)c$ for each c in C, so $cb_g = b_g g(c)$, that is, $b_g(c - g(c)) = 0$ for each $g \in G$ and $c \in C$. But C is a commutative G-Galois extension of C^G, so the ideal of C generated by $\{c - g(c) \mid c \in C\}$ is C ([4], Proposition 1.2-(5)). Thus $b_g = 0$ for each $g \neq 1$. But then $\sum_{g \in G} b_g g = b_1 \in B$. Hence $V_{B*G}(C) \subseteq B$, and so $V_{B*G}(C) = B$.

Theorem 3.2. Assume C is a G-Galois algebra over C^G with Galois group $G|_C \cong G$. Then the following statements are equivalent:
(1) $B * G$ is Azumaya.
(2) $(B * G)^{\bar{G}}$ is Azumaya.
(3) B is Azumaya.

Proof: (1) $\Longleftrightarrow$ (2) Since C is a G-Galois algebra over C^G with Galois group $G|_C \cong G$, there exists an element $c \in C$ such that $\mathrm{Tr}_G(c) = 1$, where $\mathrm{Tr}_G(\ )$ is the trace of G ([4], Corollary 1.3-(1)). By Lemma 3.1-(b), $B * G$ is H-separable over $(B * G)^{\bar{G}}$ and $B * G$ is a finitely generated and projective left module over $(B * G)^{\bar{G}}$ ([5], Theorem1), so (2) $\Longrightarrow$ (1) by ([6], Theorem 1). Conversely, since the restriction of $\bar{G}$ to C is G, $(B * G)^{\bar{G}}$ is a direct summand of $B * G$ as a $(B * G)^{\bar{G}}$-bimodule by using the fact that $\mathrm{Tr}_G(c) = 1$. Thus the separability of $B * G$ over Z implies the separability of $(B * G)^{\bar{G}}$ over Z by the argument as given on p. 120 in [5]. Since Z is contained in the center of $(B * G)^{\bar{G}}$, $(B * G)^{\bar{G}}$ is Azumaya. This proves (1) $\Longrightarrow$ (2).

(1) $\Longrightarrow$ (3) Assume $B * G$ is Azumaya. Since C is a G-Galois algebra over C^G, $Z = C^G$ by Lemma 3.1-(c). Hence $B * G$ is an Azumaya C^G-algebra. By Lemma 3.1-(d), $V_{B*G}(C) = B$. Therefore, B is a separable C^G-algebra (for C is a separable C^G-algebra) by the commutator theroem for Azumaya algebras ([4], Theorem 4.3, p. 57). Thus B is an Azumaya algebra.

(3) $\Longrightarrow$ (1) Since C is a $G|_C$-Galois algebra over C^G, $B * G$ is a H-saparable extension of B by Lemma 3.1-(a). By hypothesis, B is an Azumaya C-algebra, so $B * G$ is a separable extension over C by the transitivity of separable extensions. Noting that C is a separable C^G-algebra (for it is G-Galois), we conclude that $B * G$ is a separable extension of C^G. Moreover, by Lemma 1-(c), $Z = C^G$, so $B * G$ is an Azumaya C^G-algebra.

Theorem 3.3. Let B be a G-Galois extension of B^G and $n^{-1} \in B$. Then, $B * G$ is an Azumaya algebra if and only if so is $(B * G)^{\bar{G}}$. In this case, the center of $(B * G)^{\bar{G}}$ is the center of ZG where Z is the center of $B * G$.

Proof: Since $n^{-1} \in B$, $\mathrm{Tr}_G(n^{-1}) = 1$. By hypothesis B is a G-Galois extension of B^G, so $B * G$ is a $\bar{G}$-Galois extension of $(B * G)^{\bar{G}}$ with an inner Galois group $\bar{G}$ with the same Galois system as B. Thus the argument in the proof of (1) $\Longleftrightarrow$ (2) in Theorem 3.2 implies that $B * G$ is an Azumaya algebra if and only if so is $(B * G)^{\bar{G}}$.

Next, we calculate the center of $(B * G)^{\bar{G}}$. Let Z be the center of $B * G$. Then the center of $(B * G)^{\bar{G}} = V_{(B*G)^{\bar{G}}}((B * G)^{\bar{G}}) = (B * G)^{\bar{G}} \cap V_{B*G}((B * G)^{\bar{G}}) =$ $(B * G)^{\bar{G}} \cap V_{B*G}(V_{B*G}(ZG))$. Since $n^{-1} \in B$, ZG is a separable Z-algebra. Hence $V_{B*G}(V_{B*G}(ZG)) = ZG$ because $B * G$ is an Azumaya Z-algebra ([4], Theorem 4.3, p. 57). Thus, the center of $(B*G)^{\bar{G}} = (B*G)^{\bar{G}} \cap (ZG) = V_{B*G}(ZG) \cap (ZG) = V_{ZG}(ZG) =$ the center of ZG.

4 Splitting Rings

In this section, we shall show that some splitting rings for $B * G$, $(B * G)^{\bar{G}}$ and B are the same. By Theorem 5.5 in [4] on p. 64, for an Azumaya algebra A, any separable maximal commutative subalgebra of A is a splitting ring for A. In this section, we call F a splitting ring for A if F is a separable maximal commutative subalgebra of A. We first give a result on the splitting rings for any Azumaya algebra.

Theorem 4.1. Let A be an Azumaya C-algebra and D a separable commutative subalgebra of A. Then, (i) $V_A(D)$ is an Azumaya D-algebra, and (ii) F is a splitting ring for A containing D if and only if F is a splitting ring for $V_A(D)$ over D.

Proof: (i) Since A is an Azumaya C-algebra and D a separable subalgebra of A, $V_A(V_A(D)) = D$ and $V_A(D)$ is separable subalgebra of A by the commutator theorem for Azumaya algebras ([4], Theorem 4.3, p. 57). Since D is a commutative subalgebra of A, $C \subset D \subset$ the center of $V_A(D)$. Hence $V_A(D)$ is separable over D. Moreover, the center of $V_A(D) = V_{V_A(D)}(V_A(D)) \subset V_A(V_A(D)) = D$; and so the center of $V_A(D) = D$, that is, $V_A(D)$ is an Azumaya D-algebra.

(ii) ($\Longrightarrow$) Let F be a splitting ring for A containing D. Then $D \subset F$ and $F = V_A(F)$, and so $F = V_A(F) \subset V_A(D)$. Hence $V_{V_A(D)}(F) = V_A(D) \cap V_A(F) = V_A(F) = F$. Thus F is a maximal commutative subalgebra of $V_A(D)$. Moreover, since F is separable over C and $C \subset$ the center of $V_A(D) = D \subset F =$ the center of F, F is separable over D. Thus, F is splitting ring for $V_A(D)$ over D.

($\Longleftarrow$) Let F be splitting ring for $V_A(D)$ over D. Then $D \subset F$ and $F = V_{V_A(D)}(F)$, and so $V_A(F) \subset V_A(D)$. Hence $V_A(F) = V_A(D) \cap V_A(F) = V_{V_A(D)}(F) = F$. Thus F is a maximal commutative subalgebra of A. Moreover, since F is is separable over D and D is separable over C, F is separable over C. Therefore, F is splitting ring for A.

Theorem 4.2. Assume B is a DeMeyer-Kanzaki G-Galois extension (that is, B is an Azumaya C-algebra and C is a G-Galois extension of C^G with $G|_C \cong G$). Then, F is a splitting ring for the Azumaya algebra $B * G$ containing C if and only if F is a splitting ring for the Azumaya algebra B.

Proof: ($\Longrightarrow$) Assume F is a splitting ring for the Azumaya algebra $B * G$ containing C. Then $C \subseteq F$ and $F = V_{B*G}(F)$. Hence $F = V_{B*G}(F) \subseteq V_{B*G}(C)$. Since C is a

G-Galois extension of C^G, $V_{B*G}(C) = B$ by Lemma 3.2-(d). Thus $V_{B*G}(F) \subseteq V_{B*G}(C) = B$. Therefore $V_{B*G}(F) = V_B(F)$. But then $F = V_{B*G}(F) = V_B(F)$; and so F is a splitting ring for B.

($\Longleftarrow$) Let F be a splitting ring for the Azumaya algebra B. Then $C \subseteq F$ and $F = V_B(F)$. Hence $V_{B*G}(F) \subseteq V_{B*G}(C)$. By Lemma 3.2-(d) again, $V_{B*G}(C) = B$, so $V_{B*G}(F) \subseteq V_{B*G}(C) = B$. Thus $V_{B*G}(F) = V_B(F)$; and so $F = V_B(F) = V_{B*G}(F)$. Therefore, F is a splitting ring for the Azumaya algebra $B * G$ containing C.

Next, we consider another Galois condition on B.

Theorem 4.3. Let B be a G-Galois extension of B^G, $n^{-1} \in B$ and $B * G$ an Azumaya algebra. Then, F is a splitting ring for $B * G$ containing D, where D is the center of $(B * G)^{\bar{G}}$ if and only if F is a splitting ring for $V_{B*G}(D)$.

Proof: This is an immediate consequence of Theorem 4.1-(ii) for the Azumaya algebra $B * G$.

Corollary 4.4. Assume B is a G-Galois extension of B^G, $n^{-1} \in B$ and $B * G$ an Azumaya algebra. Let G be an Abelian group. Then, F is a splitting ring for $B * G$ containing ZG if and only if F is a splitting ring for $(B * G)^{\bar{G}}$.

Proof: Since G is Abelian, $n^{-1} \in B$ and Z is the center of $B*G$, ZG is a commutative separable subalgebra. Let $D = ZG$. Then D is the center of $(B * G)^{\bar{G}}$ by Theorem 3.4. Moreover, $V_{B*G}(D) = V_{B*G}(ZG) = (B * G)^{\bar{G}}$, so by Theorem 4.3, F is a splitting ring for $B * G$ containing $ZG(= D)$ if and only if F is a splitting ring for $(B * G)^{\bar{G}}(= V_{B*G}(D))$.

Under the hypothesis of Theorem 4.3, for a splitting ring for $B * G$ containing the center of $(B * G)^{\bar{G}}$, we shall show that F splits $(B * G)^{\bar{G}}$ is equivalent to that G is Abelian.

Theorem 4.5. Assume B is a G-Galois extension of B^G, $n^{-1} \in B$ and $B * G$ is Azumaya algebra. Let F be a splitting ring for $B * G$ containing D, where D is the center of $(B * G)^{\bar{G}}$. Then, F is a splitting ring for $(B * G)^{\bar{G}}$ if and only if G is Abelian.

Proof: ($\Longrightarrow$) Since F is a splitting ring for $B * G$, $F = V_{B*G}(F)$. Now, $F = V_{(B*G)^{\bar{G}}}(F)$, so $F = V_{(B*G)^{\bar{G}}}(F) = (B*G)^{\bar{G}} \cap V_{B*G}(F) = (B*G)^{\bar{G}} \cap F$. Thus $F \subset (B*G)^{\bar{G}}$, and so $F \subset V_{B*G}(ZG))$. Therefore, $V_{B*G}(V_{B*G}(ZG)) \subset V_{B*G}(F) = F$. Since $n^{-1} \in B$, ZG is a separable Z-algebra. Hence $V_{B*G}(V_{B*G}(ZG)) = ZG$ because $B*G$ is an Azumaya Z-algebra ([4], Theorem 4.3, p. 57). Thus, $ZG \subset F$. But F is commutative, so G is Abelian.

($\Longleftarrow$) Assume G is Abelian. Since Z is the center of $B*G$, ZG is commutative. Hence $ZG \subset F$, and so $F = V_{B*G}(F) \subset V_{B*G}(ZG)$. Thus $F = V_{B*G}(F) = V_{B*G}(ZG) \cap V_{B*G}(F) = (B*G)^{\bar{G}} \cap V_{B*G}(F) = V_{(B*G)^{\bar{G}}}(F)$. Therefore, F is a splitting ring for $(B*G)^{\bar{G}}$.

By Corollary 4.4 and Theorem 4.5, under the hypothesis of Theorem 4.3, two of the following statements imply the third:

(1) F is a splitting ring for $B*G$ containing the center of $(B*G)^{\bar{G}}$.

(2) F is a splitting ring for $(B*G)^{\bar{G}}$.

(3) G is Abelian.

We conclude the present paper with two examples of skew group rings $B*G$ to show the relationship of the splitting rings between $B*G$, B, and $(B*G)^{\bar{G}}$.

Example 1. Let $B = Q[i,j,k] = Q + Qi + Qj + Qk$ be the quaternion algebra over the rational field Q, $G = \{g_1 = 1, g_i, g_j, g_k | \ g_i(x) = ixi^{-1},\ g_j(x) = jxj^{-1},\ g_k(x) = kxk^{-1}$ for all $x \in B\}$, and $A = B*G$. Then

(1) B is a G-Galois extension of B^G with G-Galois system $\{\frac{1}{2}, -\frac{1}{2}i, -\frac{1}{2}j, -\frac{1}{2}k\ ;\ \frac{1}{2}, \frac{1}{2}i, \frac{1}{2}j, \frac{1}{2}k\}$ and $4^{-1} \in B$.

(2) $B^G = Q$, so A is an Azumaya Q-algebra ([1], Theorem 3.1).

(3) $D = Q[i] = Q + Qi$ is a commutative separable Q-subalgebra of A.

(4) $V_A(D) = D + Dg_i + (Qj + Qk)g_j + (Qj + Qk)g_k$ is an Azumaya D-algebra by Theorem 4.1-(i).

(5) $F = D + Dg_i$ is a splitting ring for $V_A(D)$, so, by Theorem 4.1-(ii), $F = D + Dg_i$ is also a splitting ring for A.

(6) $(B*G)^{\bar{G}} = V_{B*G}(QG) = QG$ which is a commutative separable subalgebra, so QG is a splitting ring for $(B*G)^{\bar{G}}(= QG)$ and for $B*G$ by Theorem 4.3 (or Corollary 4.4 for G is Abelian).

Example 2. Let $M_2(Q)$ be the matrix ring of order 2 over the rational field Q, $B = M_2(Q) \oplus M_2(Q)$, $g : B \longrightarrow B$ by $g(a,b) = (b,a)$ for all $(a,b) \in B$. Then,

(1) g is an automorphism of B of order 2.

(2) Let $G = \{1,\ g\}$. Then B is a G-Galois extension of B^G with the Galois system $\{a_1 = (I,0),\ a_2 = (0,I)\,;\, b_1 = (I,0),\ b_2 = (0,I)\}$, that is, $a_1b_1 + a_2b_2 = (I,I)$ and $a_1g(b_1) + a_2g(b_2) = (0,0)$, where I is the identity of $M_2(Q)$ and 0 is the zero matrix in $M_2(Q)$.

(3) Let C be the center of B. Then $C = Q \oplus Q$, and C is a G-Galois extension of C^G with the same Galois system as B and $G|_C \cong G$.

(4) $B*G$ is an Azumaya C^G-algebra where $C^G = \{(a,a)|a \in Q\}$ since B is an Azumaya C-algebra by Theorem 3.2.

(5) $(B * G)^{\bar{G}} = C^G + C^G g$.

(6) Since C is a commutative separable subalgebra of $B * G$, $V_{B*G}(C)$ is an Azumaya C-algebra by Theorem 4.1-(i).

(7) $V_{B*G}(C) = B$ by Lemma 3.1-(d).

(8) Let $F = Q[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}] + Q[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}]$. Then F is a separable maximal commutative subalgebra of $M_2(Q)$, and so $F \oplus F$ is a separable maximal commutative subalgebra of B, that is, $F \oplus F$ is a splitting ring for B. Thus, $F \oplus F$ is a splitting ring for $B * G$ by Theorem 4.2.

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