

# On Three Types of Galois Extensions of Rings

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**Abstract.** The class of the Azumaya Galois extensions of rings are generalized to the class of centrally projective Galois extensions and the class of faithfully Galois extensions. Each class is characterized and the inclusion relationships are also proved.

## 1. Introduction

A Galois extension  $S$  is called a central Galois algebra if it is Galois over its center, and a Galois extension  $S$  is called the DeMeyer-Kanzaki Galois if its center is Galois with the Galois group induced by and isomorphic with the Galois group of  $S$ . These two classes were intensively investigated in [4] and [7]. Recently, these classes were generalized to a broader class of the Azumaya Galois extensions in [1] and [2]; that is,  $S$  is a  $G$ -Galois extension of  $S^G$  which is an Azumaya  $C^G$ -algebra where  $C$  is the center of  $S$ . The purpose of this paper is to give two more bigger classes than the Azumaya Galois extensions: (i) The centrally projective Galois extensions (CP-Galois extensions), and (ii) The faithfully Galois extensions, where a CP- $G$ -Galois extension  $S$  is a  $G$ -Galois and centrally projective over  $S^G$  (that is,  $S$  is a direct summand of a finite direct sum of  $S^G$  as a  $S^G$ -bimodule), and  $S$  is a faithfully Galois extension if  $S * G$  is an Azumaya algebra and  $S$  is faithful as a left  $S * G$ -module. We shall characterize each class, and show an inclusion relationship between them. Examples are also given for each class. This work was supported by a Caterpillar Fellowship at Bradley University. We would like to thank Caterpillar Inc. for the support.

## 2. Preliminaries

Throughout, we assume that  $S$  is a ring with 1,  $G = \{g_1, g_2, \dots, g_n\}$  a finite automorphism group of  $S$  of order  $n$  for some integer  $n$  invertible in  $S$ ,  $S^G$  the subring of the elements

fixed under each element in  $G$ ,  $S * G$  a skew group ring of  $G$  over  $S$ ,  $Z$  the center of  $S * G$ , and  $G'$  the inner automorphism group of  $S * G$  induced by the elements in  $G$ , that is,  $g'(x) = gxg^{-1}$  for each  $g$  in  $G$  and  $x$  in  $S * G$ , so the restriction of  $G'$  to  $S$  is  $G$ .

Following [4], [5], and [9] we call  $S$  a  $G$ -Galois extension of  $S^G$  if there exist elements  $\{c_i, d_i$  in  $S$ ,  $i = 1, 2, \dots, k$  for some integer  $k\}$  such that  $\sum c_i g(d_i) = \delta_{1,g}$  for each  $g \in G$ . Let  $B$  be a subring of a ring  $A$  with 1.  $A$  is called a separable extension of  $B$  if there exist  $\{a_i, b_i$  in  $A$ ,  $i = 1, 2, \dots, m$  for some integer  $m\}$  such that  $\sum a_i b_i = 1$ , and  $\sum s a_i \otimes b_i = \sum a_i \otimes b_i s$  for all  $s$  in  $A$  where  $\otimes$  is over  $B$ . An Azumaya algebra is a separable extension of its center. A ring  $A$  is called a  $H$ -separable extension of  $B$  if  $A \otimes_B A$  is isomorphic to a direct summand of a finite direct sum of  $A$  as a  $A$ -bimodule (that is,  $A \otimes_B A$  is a centrally projective module over  $A$ ). It is known that an Azumaya algebra is a  $H$ -separable extension and an  $H$ -separable extension is a separable extension. A ring  $S$  is called an Azumaya  $G$ -Galois extension of  $S^G$  if it is a  $G$ -Galois extension of  $S^G$  which is a  $C^G$ -Azumaya algebra where  $C$  is the center of  $S$ . Next we give two more bigger classes than the Azumaya Galois extensions. Throughout, let  $B$  be a subring of a ring  $A$ ,  $V_A(B)$  means the commutator subring of  $B$  in  $A$ .

**Definition 1.**  $S$  is called a *centrally projective Galois extension (CP-Galois extension)* if  $S$  is  $G$ -Galois and centrally projective over  $S^G$  (that is,  $S$  is a direct summand of a finite direct sum of  $S^G$  as a  $S^G$ -bimodule).

**Definition 2.**  $S$  is called a *faithfully Galois extension* if  $S$  is faithful as a left  $S * G$ -module and  $S * G$  is an Azumaya  $Z$ -algebra.

We shall characterize each class, and show that a Galois extension is a CP-Galois extension and that a CP Galois extension of an Azumaya algebra is a faithfully Galois extension. We also give examples to illustrate the proper inclusion relationships.

### 3. Classes of Galois extensions

Keeping all notations of section 2, we shall characterize each class of Galois extensions.

**Theorem 3.1.**  $S$  is an Azumaya Galois extension if and only if  $S * G$  is a  $H$ -separable extension of  $S$  and  $S^G$  is an Azumaya  $C^G$ -algebra.

*Proof.* The necessity is a consequence of Theorem 3.1 in [1].

For the sufficiency, it suffices to show that  $S$  is a  $G$ -Galois extension of  $S^G$ . At first, since  $n$  is a unit in  $S$ ,  $S * G$  is a separable extension with a separable system  $\{\frac{1}{n}g_i, g_i^{-1} \mid i = 1, 2, \dots, n\}$ ; that is,  $\sum_{i=1}^n \frac{1}{n}g_i g_i^{-1} = 1$  and  $a \sum_{i=1}^n \frac{1}{n}g_i \otimes g_i^{-1} = \sum_{i=1}^n \frac{1}{n}g_i \otimes g_i^{-1} a$  for all  $a$  in  $S * G$  where  $\otimes$  is over  $S$ . Consider  $S$  as a left  $S * G$ -module by  $(tg)(s) = t(g(s))$  for all  $t, s \in S$  and  $g \in G$ , we can show that  $S$  is a finitely generated and projective module over  $S * G$  by using the above separable system  $\{\frac{1}{n}g_i, g_i^{-1} \mid i = 1, 2, \dots, n\}$  (see the proof of Proposition 2.3 in [5]). Moreover, since  $S * G$  is a  $H$ -separable extension of  $S$ ,  $S$  is a generator over  $S * G$  (for  $S$  is a generator over  $S$ ) ([6]). Thus  $S$  is a progenerator over  $S * G$ . Therefore, noting that  $S^G \cong \text{Hom}_{S * G}(S, S)$ , we conclude that  $S * G \cong \text{Hom}_{S^G}(S, S)$  by the Morita theorem. This implies that  $S$  is a  $G$ -Galois extension of  $S^G$  ([4], Theorem 1).

We recall that  $S$  is a CP  $G$ -Galois extension of  $S^G$  if  $S$  is a  $G$ -Galois and centrally projective over  $S^G$ . Next we characterize a CP-Galois extension.

**Theorem 3.2.**  *$S$  is a CP- $G$ -Galois extension of  $S^G$  if and only if  $S * G$  is a  $H$ -separable extension of  $S$  (for more properties of a CP-Galois, see[3]).*

*Proof.* Let  $S$  be a CP- $G$ -Galois extension of  $S^G$ . Then  $S$  is a  $G$ -Galois extension of  $S^G$ . Hence  $S$  is a finitely generated and projective module over  $S^G$ . But  $n$  is a unit in  $S$ , so  $n^{-1}tr_G(\ ) : S \rightarrow S^G \rightarrow 0$  splits the inclusion map  $S^G \rightarrow S$  as a  $S^G$ -module, where  $tr_G(\ )$  is the trace of  $G$ . Therefore,  $S$  is a progenerator over  $S^G$ . Moreover, by hypothesis,  $S$  is a centrally projective over  $S^G$ , so  $\text{Hom}_{S^G}(S, S)$  is a  $H$ -separable extension of  $S$  ([10], Proposition 11). Since  $S$  is a  $G$ -Galois extension of  $S^G$  again,  $S * G \cong \text{Hom}_{S^G}(S, S)$ ,  $S * G$  is a  $H$ -separable extension of  $S$ .

Conversely, since  $n$  is a unit in  $S$ ,  $S$  is a finitely generated and projective module as a left  $S * G$ -module as shown in Theorem 3.1. Also, that  $S * G$  is a  $H$ -separable extension of  $S$  implies that  $S$  is a generator over  $S * G$  ([6], Lemma). Thus  $S$  is a progenerator over  $S * G$ . But then  $S$  is a  $G$ -Galois extension such that  $S * G \cong \text{Hom}_{S^G}(S, S)$ ; and so  $S$  is a centrally projective  $S^G$ -module ([10], Proposition 11). Therefore,  $S$  is a CP- $G$ -Galois extension of  $S^G$ .

In [11], the class of  $F$ -Azumaya algebras were studied where  $S$  is called a  $F$ -Azumaya algebra if  $S$  is an Azumaya  $C^G$ -algebra and is faithful as a left  $S * G$ -module. Also in Theorem 3.1 ([2]), it was shown that  $S$  is an Azumaya  $G$ -Galois extension if and only if  $S * G$  is an Azumaya  $C^G$ -algebra. Next we characterize an Azumaya skew group ring  $S * G$

whose center  $Z$  is not necessarily  $C^G$ , but  $S$  is a faithful left  $S * G$ -module; that is,  $S$  is a faithfully Galois extension.

**Theorem 3.3.**  *$S * G$  is a  $G'$ -Galois extension of  $(S * G)^{G'}$  which is a separable  $Z$ -algebra if and only if  $S$  is a faithfully Galois extension of  $S^G$ , that is,  $S * G$  is an Azumaya  $Z$ -algebra and  $S$  is a faithful left  $S * G$ -module.*

*Proof.* Let  $S * G$  be a  $G'$ -Galois extension of  $(S * G)^{G'}$ . Then it is well known that  $S * G$  is a separable extension of  $(S * G)^{G'}$ . But  $(S * G)^{G'}$  is a separable  $Z$ -algebra by hypothesis, so  $S * G$  is a separable  $Z$ -algebra by the transitivity of separable extensions, that is,  $S * G$  is an Azumaya  $Z$ -algebra. Moreover,  $S * G$  is a  $G'$ -Galois extension again, so  $S$  is a  $G$ -Galois extension ([12], Theorem 3.1). Hence  $S * G \cong \text{Hom}_{S^G}(S, S)$ . Thus  $S$  is faithful as a left  $\text{Hom}_{S^G}(S, S)$ -module. Therefore,  $S$  is faithful as a left  $S * G$ -module.

Conversely, since  $S * G$  is an Azumaya  $Z$ -algebra, it is faithful over  $Z$ . Hence  $S$  is faithful over  $Z$  by the transitivity of faithful modules (for  $S$  is faithful over  $S * G$  by hypothesis). But  $n$  is a unit in  $S$ , so  $S$  is a finitely generated and projective  $S * G$ -module by the same argument as given in the proof of Theorem 3.1. Noting that  $S * G$  is an Azumaya  $Z$ -algebra again, we have that  $S$  is a finitely generated and projective  $Z$ -module by the transitivity of finitely generated and projective modules. Now,  $Z$  is a commutative ring, so  $S$  is a generator over  $Z$  (for it is faithful). But then  $S$  is a progenerator over  $S * G$  (for  $S * G$  is an Azumaya algebra). Therefore  $S$  is a  $G$ -Galois extension of  $S^G$ . This implies that  $S * G$  is a  $G'$ -Galois extension of  $(S * G)^{G'}$  with the same Galois system as  $S$  where  $G'$  is the inner automorphism group of  $S * G$  induced by  $G$ . Moreover, since  $n$  is a unit in  $Z$ ,  $ZG$  is a separable  $Z$ -algebra with a separable system  $\{\frac{1}{n}g_i, g_i^{-1} \mid i = 1, 2, \dots, n\}$ . Since  $S * G$  is an Azumaya  $Z$ -algebra by hypothesis,  $(S * G)^{G'} (= V_{S * G}(ZG))$  is also a separable  $Z$ -algebra by the commutator theorem for Azumaya algebras ([5], Theorem 4.3, P.57).

**Corollary 3.4.**  *$S$  is a faithfully Galois extension of  $S^G$  if and only if  $S$  is a Galois extension of  $S^G$  and  $(S * G)^{G'}$  is a separable  $Z$ -algebra.*

*Proof.* The Corollary is immediate by Theorem 3.1 in [12].

#### 4. The Inclusion Relationship

In this section we shall show an inclusion relationship between the classes of Galois extensions as given in section 3, and give an example in each class.

**Theorem 4.1.** *Any Azumaya Galois extension is a CP-Galois extension.*

*Proof.* Let  $S$  be an Azumaya Galois extension. Then  $S * G$  is a  $H$ -separable extension of  $S$  ([1], Theorem 3.1). Thus  $S$  is an CP-Galois extension by Theorem 3.2.

**Theorem 4.2.** *Any CP- $G$ -Galois extension  $S$  of an Azumaya algebra (that is,  $S^G$  is an Azumaya algebra) is a faithfully Galois extension.*

*Proof.* Let  $S$  be a CP- $G$ -Galois extension of  $S^G$  which is Azumaya. Then  $S$  is a  $G$ -Galois extension. Hence  $S$  is a faithful left  $\text{Hom}_{S^G}(S, S)$ -module, and therefore,  $S$  is a faithful left  $S * G$ -module. Moreover,  $S$  is a centrally projective module over  $S^G$  by hypothesis, so  $\text{Hom}_{S^G}(S, S)$  is a projective  $H$ -separable extension of  $S^G$  ([10], Theorem 6). But  $S^G$  is an Azumaya algebra by hypothesis, so  $\text{Hom}_{S^G}(S, S)$  is an Azumaya algebra ([8], Theorem 1). Thus  $S * G$  is also an Azumaya algebra. Therefore,  $S$  is a faithfully Galois extension.

In the following, we give four examples: (i)  $S$  is an Azumaya Galois extension, (ii)  $S$  is a CP-Galois extension, but not an Azumaya Galois extension, (iii)  $S$  is a faithfully Galois extension, but not a CP-Galois extension, and (iv)  $S$  is a CP-Galois extension, but not a faithfully Galois extension.

*Example 1.* Let  $Q[i, j, k]$  be the quaternion algebra over the rational field  $Q$  and  $G = \{1, g_i, g_j, g_k \mid g_i(x) = xix^{-1}, g_j(x) = xjx^{-1}, g_k(x) = kxk^{-1} \text{ for all } x \text{ in } Q[i, j, k]\}$ . Then

(1)  $Q[i, j, k]$  is a central  $G$ -Galois algebra over  $Q$  with a Galois system  $\{\frac{1}{2}, \frac{1}{2}i, \frac{1}{2}j, \frac{1}{2}k; \frac{1}{2}, -\frac{1}{2}i, -\frac{1}{2}j, -\frac{1}{2}k\}$ .

(2) Let  $M_2(Q)$  be the matrix algebra of order 2 over  $Q$ . Then  $S = M_2(Q) \otimes_Q Q[i, j, k]$  is a  $1 \times G$ -Galois extension of  $M_2(Q)$  which is an Azumaya algebra over  $Q$ ; that is,  $S$  is an Azumaya Galois extension of  $M_2(Q)$ .

*Example 2.* Let  $A = Q[i, j, k]$  as given in Example 1, and  $B$  a non-Azumaya  $Q$ -algebra. Then

(1)  $S = B \otimes_Q A$  is a  $1 \times G$ -Galois extension of  $B$ , where  $G$  is given in Example 1, for  $A$  is a  $G$ -Galois extension of  $Q$ .

(2)  $S$  is a centrally projective  $B$ -module for  $A$  is a centrally projective  $Q$ -module.

(3)  $S$  is a CP-Galois extension of  $B$  by (1) and (2), but  $B$  is not an Azumaya  $Q$ -algebra.

Thus  $S$  is not an Azumaya Galois extension.

*Example 3.* Let  $S = Q[i, j, k]$  be the quaternion algebra over the rational field  $Q$  and  $G = \{1, g_i \mid g_i(x) = xixi^{-1} \text{ for all } x \text{ in } Q[i, j, k]\}$ . Then

- (1)  $S$  is a  $G$ -Galois extension of  $S^G$  with  $G$ -Galois system  $\{\frac{1}{2}, -\frac{1}{2}i, -\frac{1}{2}j, -\frac{1}{2}k\}$ ;  $\{\frac{1}{2}, \frac{1}{2}i, \frac{1}{2}j, \frac{1}{2}k\}$ .
- (2)  $S^G = Q[i]$  is a commutative separable  $Q$ -algebra.
- (3) The center of  $S$  is  $Q = C^G$ , so  $S^G$  is not an Azumaya  $C^G$ -algebra. Thus  $S$  is not an Azumaya  $G$ -Galois extension.
- (4)  $S * G$  is an Azumaya  $Z$ -algebra for  $S * G$  is a separable  $Q$ -algebra and  $Q \subset Z$ .
- (5)  $(S * G)^{G'} = V_{S * G}(ZG)$  is a separable  $Z$ -algebra for  $ZG$  is a separable  $Z$ -algebra.
- (6)  $S$  is a faithfully Galois extension by Theorem 3.3.
- (7)  $S * G$  is not a  $H$ -separable extension of  $S$  for the center  $Z$  of  $S * G$  is not  $C^G (= Q)$  (see [1], Theorem 3.1). Thus  $S$  is not a CP-Galois extension by Theorem 3.3.

*Example 4.* Let  $A = Q[i, j, k]$ ,  $G = \{1, g_i, g_j, g_k\}$  as given in Example 2, and  $B$  a  $Q$ -algebra but not Azumaya over its center. Then

- (1)  $S = B \otimes_Q A$  is a CP-Galois extension of  $S^G (= B)$  by Example 2.
- (2)  $S$  is not a faithfully Galois extension of  $S^G$ . Suppose  $S$  is a faithfully Galois extension of  $S^G (= B)$ . Then  $S * G$  is an Azumaya  $Z$ -algebra by Theorem 3.3, and  $S$  is a progenerator  $Z$ -module (see the proof of Theorem 3.3). Thus  $Hom_Z(S, S)$  is an Azumaya  $Z$ -algebra. But  $S^G \cong Hom_{S * G}(S, S) \cong V_{Hom_Z(S, S)}(S * G)$ , So  $V_{Hom_Z(S, S)}(S * G)$  is an Azumaya  $Z$ -algebra by the commutator theorem for Azumaya algebras ([5], Theorem 4.3, P.57). Therefore,  $S^G (= B)$  is an Azumaya algebra. This is a contradiction.

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