The Generalized Center Galois Extensions of Rings*

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Abstract. Let B be a ring with 1, C the center of B, G a finite automorphism group of B, and $I_i = \{c - g_i(c) \mid c \in C\}$ for each $g_i \in G$. Then, B is called a center Galois extension with Galois group G if $BI_i = B$ for each $g_i \neq 1$ in G, and a weak center Galois extension with group G if $BI_i = Be_i$ for some nonzero idempotent e_i in C for each $g_i \neq 1$ in G. When e_i is a minimal element in the Boolean algebra generated by $\{e_i \mid g_i \in G\}$, Be_i is a center Galois extension with Galois group H_i for some subgroup H_i of G. Moreover, the central Galois algebra $B(1 - e_i)$ is characterized when B is a Galois algebra with Galois group G.

Key Words and Phrases: Galois extensions, Center Galois extensions, Weak Center Galois Extensions, and Azumaya algebras.

1. Introduction

Galois extensions of rings was intensively studied in sixties and seventies [3, 4, 7, 9]. Let C be a commutative ring with 1, G a finite automorphism group of C, and C^G the set of elements in C fixed under each element in G. It is well known that C is a Galois extension of C^G with Galois group G if and only if the ideal generated by $\{c - g_i(c) \mid c \in C\}$ is C for each $g_i \neq 1$ in G [2, Proposition 1.2, page 80]. The commutative Galois extension C

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was generalized to several well known classes of noncommutative Galois extensions. Let B be a ring with 1, $G = \{g_1 = 1, g_2, \cdots, g_n\}$ an automorphism group of B of order n for some integer n, and C the center of B. In [3, 7], F. R. DeMeyer and T. Kanzaki studied the class of Galois extensions B such that B is an Azumaya C-algebra which is a Galois algebra with Galois group $G|_C \cong G$. In [9], K. Sugano investigated the class of H-separable Galois extensions B. In [1], R. Alfaro and G. Szeto studied the class of Azumaya Galois extensions B, that is, B is a Galois extension of B^G which is an Azumaya C^G-algebra. Recently, we called B a center Galois extension with Galois group G if its center C is a Galois algebra over C^G with Galois group $G|_C \cong G$ [5, 6, 10, 11, 12], and it is shown that B is a center Galois extension of B^G if and only if $BI_i = B$ for each $g_i \neq 1$ in G [12, Theorem 3.2] where $I_i = \{c - g_i(c) \mid c \in C\}$. In the present paper, we shall generalize the class of center Galois extensions to a broader class of rings. A ring B is called a weak center Galois extension with group G if $BI_i = Be_i$ for some nonzero idempotent e_i in C for each $g_i \neq 1$ in G. We note that $G|_C \cong G$ and a weak center Galois extension is not necessarily a Galois extension. The purpose of the present paper is to show a structure of a weak center Galois extension B as follows: Let S be the Boolean algebra generated by $\{e_i \,|\, g_i \in G\}$, then, (1) if e_i is a minimal element in S, there is a subgroup H_i of G such that Be_i is a center Galois extension with Galois group H_i and $H_i|_{C(1-f_i)} = \{1\}$, (2) if the set of minimal elements $\{f_i \,|\, i = 1, 2, \cdots, m\}$ in S are contained in $\{e_i \,|\, g_i \in G\}$, $B = \oplus \sum_{i=1}^m Bf_i \oplus B(1 - \vee_{i=1}^m f_i)$ where Bf_i is a center Galois extension with Galois group $H_i|_{Bf_i} \cong H_i$, $H_i|_{C(1-f_i)} = \{1\}$, and $G|_{C(1-\vee_{i=1}^m f_i)} = \{1\}$. Moreover, it will be shown that the complementary direct summand $B(1 - f_i)$ of Bf_i in B is a central Galois algebra over $C(1 - f_i)$ with Galois group $H_i|_{B(1-f_i)}$ if and only if $B(1 - f_i)$ is an Azumaya algebra and equal to $\oplus \sum_{g_j \in H_i} BJ_j(1 - f_i)$ where $J_j = \{b \in B \mid bx = g_j(x)b \text{ for all } x \in B\}$, and some results are then obtained when B is a Galois algebra with Galois group G.

2. Definitions and Notations

Throughout, B will represent a ring with 1, $G = \{g_1 = 1, g_2, \cdots, g_n\}$ an automorphism group of B of order n for some integer n, C the center of B, B^G the set of elements in B fixed under each element in G, $I_i = \{c - g_i(c) \mid c \in C\}$, $J_i = \{b \in \mathrm{B} \mid bx = g_i(x)b$ for all $x \in B\}$ for each i, and $J_i^{(A)} = \{b \in \mathrm{A} \mid bx = g_i(x)b$ for all $x \in A\}$ for a subring A of B for each i.

The ring B is called a Galois extension of B^G with Galois group G if there exists a Galois system $\{a_i, b_i$ in B, $i = 1, 2, ..., p\}$ for some integer p such that $\sum_{i=1}^{p} a_i g(b_i) = \delta_{1,g}$ for each $g \in G$, and B is called a center Galois extension of B^G if C is a Galois algebra over C^G with Galois group $G|_C \cong G$. In [12], the authors of the present paper show that B is a center Galois extension of B^G with Galois group G if and only if B has a Galois system $\{b_i \in B,\ c_i \in C,\ i = 1, 2, ..., m\}$ for some integer m. B is called a weak center Galois extension of B^G with group G if $BI_i = Be_i$ for some nonzero idempotent e_i in C for each $g_i \neq 1$ in G. Obviously, a center Galois extension with Galois group G is a weak center Galois extension, and a weak center Galois extension is not necessarily a Galois extension. Throughout, we assume that B is a weak center Galois extension of B^G with group G and $E = \{e_i | i = 2, \cdots, n\}$ are the nonzero idempotents in C for each $g_i \neq 1$ in G such that $BI_i = Be_i$.

3. Center Galois Extensions

Let S be the Boolean algebra generated by the elements in E and $M = \{f_1, f_2, \cdots, f_m\}$ all distinct non-zero minimal elements in S. Then, it is easy to see that $f_1, f_2, \cdots, f_m$ are orthogonal idempotents in C. We shall show that if $M \subset E$, then $B = \oplus \sum_{i=1}^{m} Bf_i \oplus B(1 - \vee_{i=1}^{m} f_i)$ where Bf_i is a center Galois extension with Galois group $H_i|_{Bf_i} \cong H_i$, $H_i|_{C(1-f_i)} = \{1\}$ for some subgroup H_i of G, and $G|_{C(1-\vee_{i=1}^{m} f_i)} = \{1\}$. We begin with some properties of a weak center Galois extension B with group G.

Lemma 3.1. *Let B be a weak center Galois extension with group G. Then, for each $g_i \neq 1$ in G,*

(1) $g_i(e_i) = e_i$,

(2) there exists $\{b_k e_i \in Be_i; c_k e_i \in Ce_i,\ k = 1, 2, \cdots, p\}$ such that $\sum_{k=1}^{p}(b_k e_i)(c_k e_i) = e_i$ and $\sum_{k=1}^{p}(b_k e_i) g_i(c_k e_i) = 0$, and

(3) $g_i|_{C(1-e_i)} = 1$.

Proof. (1) For any $b = \sum_{k=1}^{t} b_k(c_k - g_i(c_k)) \in BI_i = Be_i$, where $b_k \in B$ and $c_k \in C$, $k = 1, 2, ..., t$ for some integer t, we have $g_i(b) = g_i(\sum_{k=1}^{t} b_k(c_k - g_i(c_k))) = \sum_{k=1}^{t} g_i(b_k)(g_i(c_k) - g_i(g_i(c_k))) \in BI_i = Be_i$. Hence $g_i(Be_i) \subset Be_i$. Thus, g_i restricted to Be_i is an automorphism of Be_i since g_i is an automorphism of B. Therefore, $g_i(e_i) = e_i$ since e_i is the identity of Be_i.

(2) Since $BI_i = Be_i$, there exist $\{b_k \in B, c_k \in C, k = 1, 2, ..., t\}$ for some integer t such that $\sum_{k=1}^{t} b_k(c_k - g_i(c_k)) = e_i$. Therefore, $\sum_{k=1}^{t} b_k c_k = e_i + \sum_{k=1}^{t} b_k g_i(c_k)$. Let $p = t+1$, $b_p = -\sum_{k=1}^{t} b_k g_i(c_k)$, and $c_p = 1$. Then $\sum_{k=1}^{p} b_k c_k = e_i$ and $\sum_{k=1}^{p} b_k g_i(c_k) = 0$. Noting that $g_i(e_i) = e_i$ by (1), we have $\sum_{k=1}^{p}(b_k e_i)(c_k e_i) = e_i$ and $\sum_{k=1}^{p}(b_k e_i) g_i(c_k e_i) = 0$.

(3) By (1), $g_i(e_i) = e_i$. Hence for any $c \in C$, $c(1-e_i) - g_i(c(1-e_i)) = (c - g_i(c))(1-e_i) \in Ce_i \cap C(1-e_i) = \{0\}$. Thus, $g_i(c(1-e_i)) = c(1-e_i)$ for all $c \in C$. This proves that g_i restricted to $C(1-e_i)$ is an identity.

Corollary 3.2. *Assume B is a weak center Galois extension with group G and each non-identity element g_i in G has order 2. Then,*

(1) Be_i is a center Galois extension with Galois group $H_{g_i} = \{1, g_i\}$ for each $g_i \neq 1$ in G, and

(2) B is a sum of center Galois extensions if $\sum_{i=2}^{n} e_i = 1$.

Lemma 3.3. *Let B be a weak center Galois extension with group G, $f_i \in M$, and $H_i = \{g_j \in G \,|\, g_j|_{C(1-f_i)} = 1\}$. Then*

(1) H_i is a subgroup of G, and

(2) $C(1-e_j) \subset C(1-f_i)$ for each $g_j \neq 1$ in H_i.

Proof. (1) It is clear.

(2) We first claim that $e_j > f_i$ for each $g_j \neq 1$ in H_i. Suppose that $e_j \not> f_i$. Since f_i is a nonzero minimal element in S, it must be that $f_i e_j = 0$. Hence $(1-f_i)e_j = e_j$. Therefore, $e_j < 1 - f_i$. Thus $Ce_j \subset C(1-f_i)$. But $g_j|_{C(1-f_i)} = 1$, so $g_j|_{Ce_j} = 1$. By Lemma 3.1-(3), $g_j|_{C(1-e_j)} = 1$. Thus, $g_j|_C = 1$. This contradicts to $I_j \neq \{0\}$ for each $g_j \neq 1$ in G. Hence it must be that $e_j > f_i$ for each $g_j \neq 1$ in H_i. Therefore, $1-e_j < 1-f_i$, and so $C(1-e_j) \subset C(1-f_i)$ for each $g_j \neq 1$ in H_i.

Lemma 3.4. *Let B, f_i, and H_i be given in Lemma 3.3 and $\{e_{i_1}, e_{i_2}, \cdots, e_{i_t}\}$ the maximal subset of E such that $f_i = e_{i_1} e_{i_2} \cdots e_{i_t}$. Then $H_i \subset \{1, g_{i_1}, g_{i_2}, \cdots, g_{i_t}\}$*

Proof. For each $g_j \neq 1$ in H_i, we have that $e_j > f_i$ by the proof of Lemma 3.3. Hence $f_i e_j = f_i$. Therefore, $e_j \in \{e_{i_1}, e_{i_2}, \cdots, e_{i_t}\}$ by the maximality property of the set. So $g_j \in \{1, g_{i_1}, g_{i_2}, \cdots, g_{i_t}\}$.

Keeping the notations of Lemma 3.3, we now show a structure theorem for a weak center Galois extension.

Theorem 3.5. *Let B be a weak center Galois extension with group G. Then,*

(1) if $H_i \neq \{1\}$, Bf_i is a center Galois extension with Galois group $H_i|_{Bf_i} \cong H_i$ and $H_i|_{C(1-f_i)} = \{1\}$, and

(2) if $H_i \neq \{1\}$ for each i, $B = \oplus \sum_{i=1}^{m} Bf_i \oplus B(1 - \vee_{i=1}^{m} f_i)$ where Bf_i is a center Galois extension with Galois group $H_i|_{Bf_i} \cong H_i$, $H_i|_{C(1-f_i)} = \{1\}$, and $G|_{C(1-\vee_{i=1}^{m} f_i)} = \{1\}$.

Proof. For each $g_j \neq 1$ in H_i, by Lemma 3.1, there exist $\{b_k^{(j)}e_j; c_k^{(j)}e_j,\ k = 1, 2, \cdots, m_j\}$ where $b_k^{(j)} \in B$ and $c_k^{(j)} \in C$, $k = 1, 2, \cdots, m_j$ for some integer m_j such that $\sum_{k=1}^{m_j}(b_k^{(j)}e_j)(c_k^{(j)}e_j) = e_j$ and $\sum_{k=1}^{m_j}(b_k^{(j)}e_j)g_j(c_k^{(j)}e_j) = 0$. Denote the elements in H_i by $\{1, g_{j_1}, g_{j_2}, \cdots, g_{j_u}\}$ for some integer u. Let $b_{k_1,k_2,\cdots,k_u} = b_{k_1}^{(j_1)}b_{k_2}^{(j_2)}\cdots b_{k_u}^{(j_u)}f_i$ and $c_{k_1,k_2,\cdots,k_u} = c_{k_1}^{(j_1)}c_{k_2}^{(j_2)}\cdots c_{k_u}^{(j_u)}f_i$ for $k_l = 1, 2, \cdots m_{j_l}$ and $l = 1, 2, \cdots u$. Noting that $c_{k_l}^{(j_l)} \in C$, $l = 1, 2, \cdots u$, we have

$$\begin{aligned}
&\sum_{k_1=1}^{m_{j_1}}\sum_{k_2=1}^{m_{j_2}}\cdots\sum_{k_u=1}^{m_{j_u}} b_{k_1,k_2,\cdots,k_u}c_{k_1,k_2,\cdots,k_u}\\
&=\sum_{k_1=1}^{m_{j_1}}\sum_{k_2=1}^{m_{j_2}}\cdots\sum_{k_u=1}^{m_{j_u}}(b_{k_1}^{(j_1)}b_{k_2}^{(j_2)}\cdots b_{k_u}^{(j_u)}f_i)(c_{k_1}^{(j_1)}c_{k_2}^{(j_2)}\cdots c_{k_u}^{(j_u)}f_i)\\
&=\sum_{k_1=1}^{m_{j_1}}(b_{k_1}^{(j_1)}e_{j_1})(c_{k_1}^{(j_1)}e_{j_1})\sum_{k_2=1}^{m_{j_2}}(b_{k_2}^{(j_2)}e_{j_2})(c_{k_2}^{(j_2)}e_{j_2})\cdots\sum_{k_u=1}^{m_{j_u}}(b_{k_u}^{(j_u)}e_{j_u})(c_{k_u}^{(j_u)}e_{j_u})f_i\\
&=e_{j_1}e_{j_2}\cdots e_{j_u}f_i = f_i,
\end{aligned}$$

and for each $g_j \neq 1$ in H_i, noting that $g_j(f_i) = f_i$, we have

$$\begin{aligned}
&\sum_{k_1=1}^{m_{j_1}}\sum_{k_2=1}^{m_{j_2}}\cdots\sum_{k_u=1}^{m_{j_u}} b_{k_1,k_2,\cdots,k_u}g_j(c_{k_1,k_2,\cdots,k_u})\\
&=\sum_{k_1=1}^{m_{j_1}}\sum_{k_2=1}^{m_{j_2}}\cdots\sum_{k_u=1}^{m_{j_u}}(b_{k_1}^{(j_1)}b_{k_2}^{(j_2)}\cdots b_{k_u}^{(j_u)}f_i)g_j(c_{k_1}^{(j_1)}c_{k_2}^{(j_2)}\cdots c_{k_u}^{(j_u)}f_i)\\
&=\sum_{k_1=1}^{m_{j_1}}(b_{k_1}^{(j_1)}f_i)g_j(c_{k_1}^{(j_1)}f_i)\sum_{k_2=1}^{m_{j_2}}(b_{k_2}^{(j_2)}f_i)g_j(c_{k_2}^{(j_2)}f_i)\cdots\sum_{k_u=1}^{m_{j_u}}(b_{k_u}^{(j_u)}f_i)g_j(c_{k_u}^{(j_u)}f_i)\\
&=\sum_{k_1=1}^{m_{j_1}}(b_{k_1}^{(j_1)}e_{j_1}f_i)g_j(c_{k_1}^{(j_1)}e_{j_1}f_i)\sum_{k_2=1}^{m_{j_2}}(b_{k_2}^{(j_2)}e_{j_2}f_i)g_j(c_{k_2}^{(j_2)}e_{j_2}f_i)\cdots\\
&\quad\cdots\sum_{k_u=1}^{m_{j_u}}(b_{k_u}^{(j_u)}e_{j_u}f_i)g_j(c_{k_u}^{(j_u)}e_{j_u}f_i)\\
&=\sum_{k_1=1}^{m_{j_1}}(b_{k_1}^{(j_1)}e_{j_1})g_j(c_{k_1}^{(j_1)}e_{j_1})f_i\sum_{k_2=1}^{m_{j_2}}(b_{k_2}^{(j_2)}e_{j_2})g_j(c_{k_2}^{(j_2)}e_{j_2})f_i\cdots\sum_{k_u=1}^{m_{j_u}}(b_{k_u}^{(j_u)}e_{j_u})g_j(c_{k_u}^{(j_u)}e_{j_u})f_i\\
&=0.
\end{aligned}$$

Thus, $\{b_{k_1,k_2,\cdots,k_u} \in Bf_i;\ c_{k_1,k_2,\cdots,k_u} \in Cf_i,\ k_l = 1, 2, \cdots m_{j_l}$ and $l = 1, 2, \cdots u\}$ is a H_i-Galois system for Bf_i. This proves that Bf_i is a center Galois extension with Galois group $H_i|_{Bf_i} \cong H_i$ [12, Theorem 3.2].

(2) Since $f_1, f_2, \cdots, f_m$ are orthogonal idempotents in C, we have that

$$B = \oplus \sum_{i=1}^{m} Bf_i \oplus B(1 - \vee_{i=1}^{m} f_i)$$

where Bf_i is a center Galois extension with Galois group $H_i|_{Bf_i} \cong H_i$ and $H_i|_{C(1-f_i)} = \{1\}$. Since $f_1, f_2, \cdots, f_m$ are all the distinct nonzero minimal elements in S which is generated by $\{e_j \,|\, g_j \in G\}$, $\vee_{i=1}^{m} f_i = \vee_{j=1}^{n} e_j$, the identity element in S. Hence $C(1 - \vee_{i=1}^{m} f_i) = C(1 - \vee_{j=1}^{n} e_j)$. Since $1 - \vee_{j=1}^{n} e_j < 1 - e_k$ for each $g_k \in G$, $C(1 - \vee_{j=1}^{n} e_j) \subset C(1 - e_k)$ for each $g_k \in G$. Thus, $G|_{C(1-\vee_{j=1}^{n} e_j)} = \{1\}$ by Lemma 3.1-(3), that is, $G|_{C(1-\vee_{i=1}^{m} f_i)} = \{1\}$. This completes the proof.

By Lemma 3.1-(3), we obtain a condition under which $H_i \neq \{1\}$.

Corollary 3.6. *Let B be a weak center Galois extension with group G. If $M \subset E$, then $B = \oplus \sum_{i=1}^{m} Bf_i \oplus B(1 - \vee_{i=1}^{m} f_i)$ where Bf_i is a center Galois extension with Galois group $H_i|_{Bf_i} \cong H_i \neq \{1\}$, $H_i|_{C(1-f_i)} = \{1\}$, and $G|_{C(1-\vee_{i=1}^{m} f_i)} = \{1\}$.*

Proof. For any f_i, there exists an $e_j \in E$ such that $f_i = e_j$ since $M \subset E$. Hence $g_j|_{C(1-f_i)} = 1$ by Lemma 3.1-(3), that is, $g_j \in H_i$. Therefore, $H_i \neq \{1\}$ for each i, and so Theorem 3.5 implies the result.

4. Central Galois Algebras

By Theorem 3.5, assuming $H_i \neq \{1\}$, we have $B = Bf_i \oplus B(1 - f_i)$ where Bf_i is a center Galois extension with Galois group $H_i|_{Bf_i} \cong H_i$ and $H_i|_{C(1-f_i)} = \{1\}$. In this section, we shall discuss a structure of $B(1 - f_i)$, the complementary direct summand of Bf_i in B. The following theorem for a central Galois algebra as given by M. Harada [4, Theorem 1] will be employed.

Proposition 4.1. *Let B be a separable R-algebra with a finite automorphism group G. If $B = \oplus\sum_{g\in G} J_g$ such that $J_g J_{g^{-1}} = C$, then B is a central Galois algebra [4, Theorem 1].*

Lemma 4.2. *If $e^2 = e \in C$ and $g_j(e) = e$, then $J_j^{(Be)} = J_j e$, where $J_j^{(Be)} = \{b \in Be \mid bx = g_j(x)b$ for all $x \in Be\}$.*

Proof. It is clear that $J_j e \subset J_j^{(Be)}$. Conversely, for any $b \in J_j^{(Be)}$, $b = be$ and $bx = g_j(x)b$ for each $x \in Be$. Hence for any $y \in B$, $by = (be)y = b(ye) = g_j(ye)b = g_j(y)eb = g_j(y)b$. Thus, $b \in J_j$, and so $b = be \in J_j e$. Therefore, $J_j^{(Be)} = J_j e$.

Theorem 4.3. *Keeping the notations as given in Theorem 3.5, $B(1 - f_i)$ is a central Galois extension with Galois group $H_i|_{B(1-f_i)} \cong H_i$ if and only if $B(1-f_i)$ is an Azumaya algebra and equal to $\oplus\sum_{g_j\in H_i} J_j(1 - f_i)$.*

Proof. ($\Longrightarrow$) Since $B(1 - f_i)$ is a central Galois extension with Galois group $H_i|_{B(1-f_i)} \cong H_i$, it is an Azumaya algebra. Moreover, by [7, Theorem 1], $B(1 - f_i) = \oplus\sum_{g_j\in H_i} J_j^{(B(1-f_i))}$. But $J_j^{(B(1-f_i))} = J_j(1 - f_i)$ by Lemma 4.2, so $B(1 - f_i) = \oplus\sum_{g_j\in H_i} J_j(1 - f_i)$.

($\Longleftarrow$) By hypothesis, $B(1 - f_i) = \oplus\sum_{g_j\in H_i} J_j(1 - f_i)$, so $B(1 - f_i) = \oplus\sum_{g_j\in H_i} J_j^{(B(1-f_i))}$ by Lemma 4.2. Also, by the definition of H_i, $H_i|_{C(1-f_i)} = \{1\}$ where $C(1 - f_i)$ is the center of $B(1 - f_i)$, so $H_i|_{B(1-f_i)}$ is a $C(1 - f_i)$-automorphism group of $B(1 - f_i)$. By hypothesis, $B(1 - f_i)$ is an Azumaya algebra over $C(1 - f_i)$, we have that $J_g^{(B(1-f_i))} J_{g^{-1}}^{(B(1-f_i))} = C(1 - f_i)$ for each $g \in H_i$ [8, Lemma 5]. Thus, $B(1 - f_i)$ is a central Galois extension with Galois group $H_i|_{B(1-f_i)} \cong H_i$ by Proposition 4.1.

Theorem 4.3 can be applied to a Galois algebra. We first give two lemmas.

Lemma 4.4. *Let B be a Galois algebra with Galois group G, $e^2 = e \in C$, and T a subset of G such that $g_j(e) = e$ and the ideal generated by $\{ce - g_j(ce) \mid ce \in Ce\}$ in Ce is Ce for each $g_j \neq 1$ in T. Then $J_j e = \{0\}$ for each $g_j \neq 1$ in T.*

Proof. For each $b \in J_j e (= J_j^{(Be)})$ for $g_j \neq 1$ in T, we have that $b = be$ and $b(xe) = g_j(xe)b$ for all $xe \in Be$. Hence $b(ce - g_j(ce)) = 0$ for all $ce \in Ce$. But the ideal generated by $\{ce - g_j(ce) \mid ce \in Ce\}$ in Ce is Ce for each $g_j \neq 1$ in T, so $b(Ce) = \{0\}$. Therefore, $b = be = 0$. Thus, $J_j e = \{0\}$ for each $g_j \neq 1$ in T.

Lemma 4.5. *Let B be a weak center Galois extension, and f_i and H_i are given in Theorem 3.5. If $f_i = e_t \in C^G$ for some g_t in G, then*

(1) H_i is normal subgroup of G, and

(2) Bf_i is a center Galois extension with Galois group $H_i|_{Bf_i} \cong H_i \neq \{1\}$ and $H_i|_{C(1-f_i)} = \{1\}$.

Proof. Part (1) is easy to check, and Corollary 3.6 implies part (2).

Theorem 4.6. *Let B be a weak center Galois extension and $f_i = e_j \in C^G$ for some g_j in G. If B is a Galois algebra with Galois group G, then, $B(1 - f_i)$ is a central Galois algebra with Galois group $H_i|_{B(1-f_i)} \cong H_i$ if and only if $C(1 - f_i)$ is a Galois algebra with Galois group G/H_i.*

Proof. ($\Longrightarrow$) We denote $(1 - f_i)$ by e. Since B is a Galois algebra with Galois group G, there exists a G-Galois system $\{a'_t, b'_t \mid t = 1, 2, ..., m$ for some integer $m\}$ such that $\sum_{t=1}^{m} a'_t g(b'_t) = \delta_{1,g}$ for each $g \in G$. Let $a_t = a'_t e$ and $b_t = b'_t e$. Then $\sum_{t=1}^{m} a_t g(b_t) = e\delta_{1,g}$ for each $g \in G$. Since $B(1 - f_i)(= Be)$ is a central Galois algebra with Galois group H_i, $|H_i|$ (the order of H_i) is a unit in Ce [7, Proposition 5]. Let $x_t = \frac{1}{|H_i|} \sum_{h \in H_i} h(a_t)$ and $y_t = \sum_{h \in H_i} h(b_t)$. Then, x_t and y_t are invariant under each element in H_i. Hence

$x_t, y_t \in Ce$ since $(Be)^{H_i} = Ce$ by hypothesis. It is straightforward to check that $\{x_t, y_t\}$ is a G/H_i-Galois system for Ce.

($\Longleftarrow$) Since B is a Galois algebra with Galois group G, B is an Azumaya algebra. Hence $B(1-f_i)$ is also an Azumaya algebra [2, Proposition 1.11, page 46]. Therefore, by Theorem 4.3, we only need to show that $B(1-f_i) = \oplus\sum_{g_j\in H_i} J_j(1-f_i)$. Since B is a Galois algebra with Galois group G again, $B = \oplus\sum_{g_j\in G} J_j$ [7, Theorem 1]. This implies that

$$B(1-f_i) = \oplus\sum_{g_j\in G} J_j(1-f_i) = (\oplus\sum_{g_j\in H_i} J_j(1-f_i)) \oplus (\oplus\sum_{g_j\notin H_i} J_j(1-f_i)).$$

By hypothesis, $C(1-f_i)(=Ce)$ is a Galois algebra with Galois group G/H_i, so the ideal generated by $\{ce - \bar{g}_j(ce) \,|\, ce \in Ce\}$ in Ce is Ce for each $\bar{g}_j \neq 1$ in G/H_i [2, Proposition 1.2, page 80]. By noting that $\bar{g}_j(ce) = g_j(ce)$, the ideal generated by $\{ce - g_j(ce) \,|\, ce \in Ce\}$ in Ce is Ce for each $g_j \notin H_i$. Thus, $J_j(1-f_i) = J_je = \{0\}$ for each $g_j \notin H_i$ by Lemma 4.4. Therefore, $B(1-f_i) = \oplus\sum_{g_j\in H_i} J_j(1-f_i)$. This completes the proof.

We conclude the present paper with an example of a weak center Galois extension as given in Theorem 3.5.

Example Let Q be the rational field, $A = Q[i,j,k]$ the quaternion division algebra over Q, $B = A\oplus A\oplus A\oplus A\oplus A$, and $G = \{g_1 = 1, g_2, g_3, g_4 = g_2g_3\}$ such that $g_2(a_1,a_2,a_3,a_4,a_5) = (a_2,a_1,a_3,a_4,a_5)$ and $g_3(a_1,a_2,a_3,a_4,a_5) = (a_1,a_2,a_4,a_3,a_5)$ for all $(a_1,a_2,a_3,a_4,a_5) \in B$. Then,

(1) $BI_i = Be_i$ for each $g_i \neq 1$ in G, where $e_2 = (1,1,0,0,0)$, $e_3 = (0,0,1,1,0)$, and $e_4 = (1,1,1,1,0)$. Hence, B is a weak center Galois extension.

(2) B is not a center Galois extension with Galois group G since $BI_2 \neq B$. Actually, B is not a Galois extension with Galois group G since G fixes the fifth component.

(3) The Boolean algebra S generated by $E = \{e_2, e_3, e_4\}$ is $S = \{0,\ e_2,\ e_3,\ e_4\}$ and the set M of all distinct non-zero minimal elements in S is $\{f_1, f_2\}$ where $f_1 = e_2$ and $f_2 = e_3$.

(4) $H_1 = \{1, g_2\}$ and $H_2 = \{1, g_3\}$.

(5) $Bf_1 (= Be_2)$ is a center Galois extension with Galois group $H_1|_{Bf_1} \cong H_1 (= \{1, g_2\})$ in which a Galois system is $\{b_1 = (1,0,0,0,0), b_2 = (0,1,0,0,0); c_1 = (1,0,0,0,0), c_2 = (0,1,0,0,0)\}$, and $Bf_2 (= Be_3)$ is a center Galois extension with Galois group $H_2|_{Bf_2} \cong H_2 (= \{1, g_3\})$ in which a Galois system is $\{b_1 = (0,0,1,0,0), b_2 = (0,0,0,1,0);$
$c_1 = (0,0,1,0,0), c_2 = (0,0,0,1,0)\}$.

(6) $B(1 - f_1 \vee f_2) = \{(0,0,0,0,a_5) \,|\, a_5 \in A\} \subset B^G$.

(7) $B = Bf_1 \oplus Bf_2 \oplus B(1 - f_1 \vee f_2)$ such that Bf_1 and Bf_2 are center Galois extensions with Galois group $H_1|_{Bf_1} \cong H_1$ and $H_2|_{Bf_2} \cong H_2$ respectively, $H_1|_{C(1-f_1)} = \{1\}$, $H_2|_{C(1-f_2)} = \{1\}$, and $G|_{C(1-f_1 \vee f_2)} = \{1\}$.

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