ELSEVIER

Journal of Statistical Planning and
Inference 72 (1998) 121–132

journal of
statistical planning
and inference

Aliasing in 2^{n-k} fractions in the case of a separation of factors

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Abstract

When studying a response as a function of several factors, engineering reasons or other deterministic considerations often imply that interactions between certain factors do not exist. This prompts advantageous exploitation of the aliasing pattern in fractional factorial designs. The general case when factors (each at two levels) can be partitioned into two classes, with no interactions present between the two classes, is treated in this paper. © 1998 Elsevier Science B.V. All rights reserved.

AMS classification: 62K10; 05B05

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1. G-estimable interactions

One of the most common occurrences in experimental science involves the investigation of a response as a function of several variables in the presence of statistical error. Fractional factorial designs and orthogonal arrays provide a userful way of carrying out such investigations. The focus of this paper is the planning stage of the experiment. From deterministic knowledge, usually due to engineering reasons, it is often known that the interactions between certain factors are zero. Typical examples are:

(i) Blocking variables that do not interact with the main factors of the study, but may or may not interact among themselves.

(ii) Situations in which there is a natural separation of factors into two or more disjoint groups such that factors in different groups act upon the response independently (or additively).

Blocking variables mentioned in (i) separate naturally from active factors. Blocks may consist of batches of raw materials, such as metals used in an experiment, or testing chambers consisting of different pieces of similar model equipment, for example.

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[*] Corresponding author. Supported by a Fulbright grant.

0378-3758/98/\$ – see front matter
PII: S0378-3758(98)00027-5

In many instances, the blocking variables produce merely an additive influence on a response, and are therefore not interacting with the other factors. In other words, the response is simply translated up or down by a constant that one calls the effect of the block in question.

Instances that give rise to a separation of factors can be found in the glass making and paint producing industries. At the PPG Research Center near Pittsburgh experiments are carried out to develop paints with various performance properties. Typical noninteractive sets of factors are (1) the paint substrates (such as steel, aluminum, or wood), (2) the chemical pretreatments, (3) the primer or color base coat, and (4) the clear. One would not expect most measured characteristics, such as the gloss or hardness on the clear, to depend on how much pretreatment or primer is put on the panel. It seems safe to assume, therefore, that these groups of factors act independently upon the response. The objective of this paper is to optimize the use of such deterministic knowledge in obtaining estimates of low order effects free of any aliasing in 2^{n-k} fractions. A general philosophical criterion of selection of fractional factorials is proposed below. A complete study of the case when factors can be separated into two classes with no interaction between factors in different classes is then carried out. Under the assumption that certain interactions are nonexistent, the criterion we propose places primary emphasis on the unaliased estimation of the main effects, then 2-factor interactions, and so on, occasionally at the expense of reducing the resolution of the design. Although nearly all designs that we construct turn out to have maximal resolution, there are cases in which the resolution is substantially lower than what it would ordinarily be. In such cases we offer resolution 5 designs as alternatives. This reduces the unaliased estimation of low order effects but it boosts the resolution, a feature desirable especially in designs involving many factors.

Let us quickly review some basic concepts and introduce the definitions and criteria relevant to the contents of this work. If factors $F_1, F_2, \ldots, F_n$ are believed to influence the response y and each factor F_i has two levels, high and low, we denote the two values that F_i can take by 1 and -1, the former when F_i is at high level and the latter when it is at low level. The set of all combinations of the levels on the n factors $F_1, F_2, \ldots, F_n$ is denoted by S. The cardinality of S is 2^n. Each level combination x in S represents a setting (or location). The response y at the location x is denoted by y_x. Furthermore, we assume that the expected value of y_x is $\eta(x)$. Whereas the parameters to be estimated are $\{\eta(x)\colon x \in S\}$, such parameters do not offer much information about the nature of dependence of y_x upon the factors $F_1, F_2, \ldots, F_n$. Instead, certain linear functions of the $\eta(x)$'s prove much more adept for such a task. The functions in questions are called interactions.

The *r-factor interaction* of $F_{i_1}, F_{i_2}, \ldots, F_{i_r}$ is $\sum_{x\in S} x_{i_1}, x_{i_2} \cdots x_{i_r} \eta(x)$. An interaction is of *order* r if it involves r distinct factors. For convenience, we denote $\sum_{x\in S} \eta(x)$ by I and the r-factor interaction of $F_{i_1}, F_{i_2}, \ldots, F_{i_r}$ by $i_1 i_2 \cdots i_r$. We call I the identity interaction and its order is 0. The interactions are parameters that are subject to estimation from data.

Fractional factorial designs are introduced to at least in part address the issue of cost. The 2^{n-k} fractional factorial designs are studied in Box and Hunter (1961a, b), Srivastava (1978), Rao (1946, 1947, 1950), and Bose (1947). When we have fewer observations than parameters not all the parameters can be estimated. To deal with this difficulty, usually the higher-order interactions are either assumed negligible, or a study of aliasing among the interactions is undertaken.

Let H be a subgroup of rank k of the full group of interactions. The elements in H are said to be *aliased* with the identity I. The full interaction group T is therefore partitioned into 2^{n-k} *aliasing patterns* (or cosets of H). Two interactions are aliased (or in the same coset of H) if and only if their product is contained in H. A 2^{n-k} *fractional factorial design* is thus formed. H is called the *subgroup of defining relations* and a set of generators of H is called its *defining relations*. For convenience, we write $A=B$ if the interactions A and B are aliased. The smallest order among the non-identity interactions in H is called the *resolution* of the design.

Denote by $l(h)$ the order of interaction h. For a subgroup of defining relations H, let $n_i(H)$ be the number of interactions of order i in H and the vector

$$V(H)=(n_1(H),n_2(H),\ldots)$$

be its *order pattern*. We say that H_1 has lesser *aberration* than H_2 if the first unequal components, say the ith components, of $V(H_1)$ and $V(H_2)$ satisfy $n_i(H_1)<n_i(H_2)$. A subgroup of defining relations H is said to have *minimum aberration* if there is no subgroup of defining relations which has lesser aberration than H. Note that the minimum aberration design has maximum resolution. A study of design aberration appears in Chen and Wu (1991), Franklin (1984) and Fries and Hunter (1980). The notion can extend to the combinatorial arrays of Rao (1973).

The *projection* of an interaction h onto factors $\{i_1,i_2,\ldots,i_r\}$ is the interaction obtained from h by ignoring the factors not from the set $\{i_1,i_2,\ldots,i_r\}$. For example, the projection of the interaction 125679 onto factors $\{1,2,3,4,5\}$ is 125 and the projection of the interaction 125679 onto factors $\{6,7,8,9\}$ is 679. The projection of a subgroup of defining relations H onto factors $\{i_1,i_2,\ldots,i_r\}$ is the subgroup obtained by projecting each interaction of H onto $\{i_1,i_2,\ldots i_r\}$. We refer to Raktoe et al. (1981) for a comprehensive exposition on factorial plans.

As was mentioned in the introductory passages, it is often known that certain interactions between factors are zero. A graph G is therefore constructed as follows. Vertices are the factors. We join vertices i and j by an edge if interactions which involve both the factors F_i and F_j are zero. To amplify, if there is an edge between two vertices (i.e., factors) then the 2-factor as well as all the higher-order interactions involving the two factors in question are assumed to be zero. Emphasis on identifying important interactions prior to experimentation, and techniques to estimate them, are also found in Hedayat and Pesotan (1992).

Given a 2^{n-k} fractional factorial design D and a graph G associated with D in the manner described above, we define the G-patterns and G-estimable interactions under the pair (D,G). The *G-patterns* under (D,G) are obtained by deleting the prescribed zero interactions from the aliasing patterns of D. An interaction is *G-estimable* under (D,G) if it is either a prescribed zero interaction or the only interaction in a G-pattern.

We wish to distinguish between the fractional factorial design (D,G) in which interactions containing edges of G are all zero, and the fractional factorial design D with no restrictions attached. To emphasize the difference we call D the *unconstrained* fractional factorial design.

For a fractional factorial design D and a graph G we define a $(n+1)$-dimensional vector,

$$m(D,G)=(m_1,m_2,\ldots,m_n;m_{n+1}),$$

where the ith coordinate m_i (written also as $m_i(D,G)$); $i=1,2,\ldots,n$, represents the number of G-estimable non-zero i-factor interactions, and m_{n+1} is the resolution of the design D.

Let D_1 and D_2 be 2^{n-k} fractional factorial designs and G a graph. We say that the design D_1 is *G-better* (or simply *better*) than D_2 if $m_1(D_1,G)>m_1(D_2,G)$ or there exists a positive integer t such that $m_i(D_1,G)=m_i(D_2,G)$ for $1\leqslant i\leqslant t$ and $m_{t+1}(D_1,G)>m_{t+1}(D_2,G)$; $1\leqslant t\leqslant n$. A 2^{n-k} fractional factorial design D is *G-best* (or simply *best*) for a given graph G if no other 2^{n-k} fractional factorial design is better than D. Subject to the existing zero interactions, the criterion is intuitive in the sense that it deems a design best if it allows the estimation of a maximal number of main effects; if there is more than one such design, then one further selects among these that design which estimates a maximal number of 2-factor interactions, and so on.

The criterion appears reasonable insofar as estimability of low order effects is concerned. However, in order to achieve maximal G-estimability we sometime sacrifice resolution, in the sense that the effects that are not G-estimable may be aliased in a rather unpleasant fashion that yields a surprisingly small resolution for the design overall. We correct this difficulty whenever it arises by providing alternative designs of resolution at least five.

As was mentioned in the beginning of this section, there are instances when the factors can be partitioned into two classes such that there are no interactions between the factors in different classes but interactions may occur among factors in the same class. We therefore call a graph *complete bipartite* if its vertices can be partitioned into two classes such that there are no edges between vertices in the same class but there is an edge between any pair of vertices from different classes.

For ease of exposition, we defer most deductive arguments to the appendix.

2. Best fractions in the presence of a bipartition

In this section we assume that there is a separation of the factors $F_1,F_2,\ldots,F_n$ into *two* classes such that there are no interactions between the factors from different classes but there may be interactions between the factors in the same class. Without loss of generality, by relabeling the factors, we can assume the partition of the factors to be $\{F_1, F_2, \ldots, F_m\}$ and $\{F_{m+1}, \ldots, F_n\}$ with $m \leqslant n-m$. Therefore, the graph G described in Section 1 is a complete bipartite graph with the partition of vertices $\{1, 2, \ldots, m\}$ and $\{m+1, \ldots, n\}$. We find the best fractional factorial design with associated graph G. Due to the methods of finding the best designs for different values of m, we divide this section into three subsections. In Section 2.1 the complete bipartite graph G has bipartition of vertices $\{1\}$ and $\{2, 3, \ldots, n\}$. In Section 2.2 the bipartition is $\{1, 2\}$ and $\{3, 4, \ldots, n\}$. Finally, in Section 2.3 the bipartition is $\{1, 2, \ldots, m\}$ and $\{m+1, m+2, \ldots, n\}$ with $2 < m \leqslant n-m$.

2.1. *Bipartition $\{1\}$ and $\{2, 3, \ldots, n\}$*

This is the case when one factor acts independent of the remaining factors. In particular, the factor in question (which we label 1) may be a blocking factor. Results, sorted by size of the fraction, are given below.

2.1.1. *The best 2^{n-1} fractional factorial design (D, G) of resolution n has defining relation $I = 12 \cdots n$. Under the design (D, G) all the interactions are estimable, except the main effect* 1 *and the interaction* $23 \cdots n$.

(The G-best design, in strict keeping with our criterion, is the design $I = 1$. In this design all interactions except 1 are G-estimable. Interest in it is limited due to the small resolution.)

2.1.2. *The best 2^{n-2} fractional factorial design (D, G) has defining relations*

$$I = B_1B_2 = B_2B_3 = B_1B_3,$$

where

$$B_i = \begin{cases} (im-m+2)(im-m+3)\cdots(im-m+(m+1))(3m+1+i), & \text{if } i \leqslant r; \\ (im-m+2)(im-m+3)\cdots(im-m+(m+1)), & \text{otherwise}, \end{cases}$$

and $n-1 = 3m+r$, with $0 \leqslant r \leqslant 2$. Under the design (D, G) the only estimable interaction is the main effect 1.

2.1.3. *The best 2^{n-k} $(k>2)$ fractional factorial design (D, G) is the design which has the same subgroup of defining relations H as the maximum resolution $2^{(n-1)-k}$ unconstrained fractional factorial design on factors $2, 3, \ldots, n$. Under (D, G), the only G-estimable non-zero interaction is the main effect* 1.

We refer to Addelman (1963) and Daniel (1962) for a construction of unconstrained fractional factorial designs of maximum resolution.

2.2. *Bipartition* $\{1,2\}$ *and* $\{3,4,\ldots,n\}$ *with* $n-2\geqslant 2$

The next case to consider, in our exhaustive enumeration, involves two factors that act independent from the rest (but not necessarily independent of one another).

The best 2^{4-2} *fractional factorial design* (D,G) *has defining relation* $I=123=124=34$. *Under* (D,G) *the main effects* 1 *and* 2 are *G-estimable.*

Let G be a bipartite graph with bipartition $\{1,2\}$ and $\{3,4,\ldots,n\}$ with $n-2>2$.

2.2.1. *The best* 2^{n-1} *fractional factorial design* (D,G) *has defining relation* $I=12\cdots n$ *and has the property that all interactions are G-estimable, except* 12 *and* $34\cdots n$.

2.2.2. *The best* 2^{n-2} *fractional factorial design has subgroup of defining relations*

$$H=\{I,12,134\cdots n,234\cdots n\},$$

and has property that the only interactions that are not G-estimable are the main effects 1 *and* 2 *and the interactions* 12 *and* $34\cdots n$.

In case 2.2.2 the resolution of the best design is 2, no matter how many factors there are. This happens because we do not assume that the higher-order interactions are necessarily zero. Whether the higher-order interactions can be assumed nonexistent, as a general rule, is a debatable point. As a compromise, if we further assume that all the 4-factor interactions are zero, then the best design ($n>7$) has defining relations:

$$\begin{aligned} I &= 134\cdots(2q+2)(3q+2+1)(3q+2+r)\\ &= 2(q+3)\cdots(3q+2)(3q+2+1)(3q+2+r)\\ &= 1234\cdots(q+2)(2q+3)\cdots(3q+2), \end{aligned}$$

where $n-2=3q+r$ with $0\leqslant r\leqslant 2$. The resolution of these alternative designs is at least 5.

In addition, when $n=7$ the best 2^{7-2} design has defining relations

$$I=12345=1267=34567.$$

In this 2^{7-2} design the only G-estimable non-zero interactions are the main effects. This is a design of resolution 4.

2.2.3. *The best* 2^{n-k} $(k\geqslant 3)$ *fractional factorial design* (D,G) *is the design whose subgroup of defining relations is the same as the subgroup of defining relations of an unconstrained maximum resolution* $2^{(n-2)-k}$ *fractional factorial design on the factors* $3,4,\ldots,n$. *The only G-estimable interactions under* (D,G) *are the main effects* 1 *and* 2 *and the 2-factor interaction* 12.

Example 2.1. We illustrate the contexts of statement 2.2.3 with an example. For $n=8$ and $k=3$, the best 2^{8-3} fractional factorial design (D,G) has subgroup of defining relations

$$H=\{I,3456,348,357,3678,4578,467,568\}.$$

The G-patterns are

$$\begin{aligned}
&H_G=\{I,3456,348,357,3678,4578,467,568\},\\
&1H_G=\{1\},\\
&2H_G=\{2\},\\
&12H_G=\{12\},\\
&3H_G=\{3,456,48,57,678,34578,3467,3568\},\\
&4H_G=\{4,356,38,3457,34678,578,67,4568\},\\
&5H_G=\{5,346,3458,37,35678,478,4567,68\},\\
&6H_G=\{6,345,3468,3567,378,45678,47,58\},\\
&7H_G=\{7,34567,3478,35,368,458,46,5678\},\\
&8H_G=\{8,34568,34,3578,367,457,4678,56\},\\
&36H_G=\{36,45,468,567,78,345678,347,358\}.
\end{aligned}$$

Therefore, the only G-estimable interactions under (D,G) are the main effects 1 and 2 and the 2-factor interaction 12. This is in agreement with the general statement made in 2.2.3.

The resolution of the design presented above is 3. It may be tempting to increase the resolution to 4 and investigate what takes place. Two resolution 4 designs that are closest competitors are presented below.

The design D_1 with subgroup of defining relations

$$H=\{I,3456,12348,12357,3678,4578,12467,12568\}.$$

allows only the G-estimation of main effects 1 and 2.

On the other hand, the design D_0 with subgroup of defining relations

$$H=\{I,3456,1348,2357,3678,4578,1467,2568\}$$

has the property that the only non-zero interaction which is G-estimable is 12. Both of these resolution 4 designs allow, therefore, fewer low order effects to be G-estimable than the resolution 3 design discussed above. A decrease from 4 to 3 in the resolution is compensated by the G-estimability of main effects 1 and 2, as well as their interaction 12.

2.3. *Bipartition* $\{1,2,\ldots,m\}$ *and* $\{m+1,m+2,\ldots,n\}$ *with* $2<m\leqslant n-m$

The generic case that we now treat involves factors $1,2,\ldots,m$, with m at least 3, that act independent of the remaining factors.

2.3.1. *The best* 2^{n-1} *fractional factorial design* (D,G) *has defining relation* $I=12\cdots n$ *and has the property that all the interactions are G-estimable, except* $12\cdots m$ *and* $(m+1)(m+2)\cdots n$.

2.3.2. Let $H_1=\{I,h_1,h_2,h_3\}$ be the subgroup of defining relations of a 2^{m-2} unconstrained minimum aberration fractional factorial design on $\{1,2,\ldots,m\}$ with $l(h_1)\leqslant l(h_2)\leqslant l(h_3)$. Let also $H_1'=\{I,h_1',h_2',h_3'\}$ be the subgroup of defining relations of a $2^{(n-m)-2}$ unconstrained minimum aberration fractional factorial design on $\{m+1, m+2,\ldots,n\}$ with $l(h_1')\geqslant l(h_2')\geqslant l(h_3')$.

The best 2^{n-2} *fractional factorial design* (D,G) *has subgroup of defining relations*

$$H=\{I,h_1h_1',h_2h_2',h_3h_3'\}$$

and has the property that the only interactions that are not G-estimable are $h_1,h_2,h_3,h_1',h_2',h_3'$.

2.3.3. Let $A_{n,k}$ be the set of all subgroups of defining relations K satisfying the conditions that (1) K is the subgroup of defining relations of a 2^{n-k} fractional factorial design on factors $\{1,2,\ldots,n\}$; (2) the projection of K onto $\{1,2,\ldots,m\}$ is the subgroup of defining relations of an unconstrained minimum aberration 2^{m-k} fractional factorial design on $\{1,2,\ldots,m\}$; (3) the projection of K onto $\{m+1,m+2,\ldots,n\}$ is the subgroup of defining relations of an unconstrained minimum aberration $2^{(n-m)-k}$ fractional factorial design on $\{m+1,m+2,\ldots,n\}$. Let $H=\{I,h_1h_1',h_2h_2',\ldots,h_{2^k-1}h_{2^k-1}'\}$ be a member of $A_{n,k}$ which has maximum resolution among all subgroups of defining relations in $A_{n,k}$, where $H_1=\{I,h_1,h_2,\ldots,h_{2^k-1}\}$ is the projection of H onto $\{1,2,\ldots,m\}$ and $H_2=\{I,h_1',h_2',\ldots,h_{2^k-1}'\}$ is the projection of H onto $\{m+1,m+2,\ldots,n\}$.

The best 2^{n-k} $(2<k<m)$ *fractional factorial design is the design* (D,G) *with subgroup of defining relations* H. *Under* (D,G) *the only interactions that are not G-estimable are the interactions*

$$h_1,h_2,\ldots,h_{2^k-1},h_1',h_2',\ldots,h_{2^k-1}'.$$

2.3.4. *The best* 2^{n-m} *fractional factorial design* (D,G) *has subgroup of defining relations* H *generated by*

$$\{1(m+1)(m+2)\cdots n,\ 2(m+1)(m+2)\cdots n,\ldots,m(m+1)(m+2)\cdots n\}.$$

Under (D, G) *all the interactions on the factors* $m+1, m+2, \ldots, n$ *are G-estimable, except the interaction* $(m+1)(m+2)\cdots n$.

In analogy to the discussion of case 2.2.2, the resolution of the best design for case 2.3.4 is 2. If we further assume that all the 4-factor interactions are zero, then we have the following alternative designs of resolution 5 or more.

If $n-m \geqslant 2k$ *and* $k>3$, *the best design D has subgroup of defining relations H generated by*

$$\{1h_1, 2h_2, \ldots, mh_m\},$$

where $\{h_1, h_2, \ldots, h_m\}$ *is a set of generators of a maximum resolution* $2^{(n-m)-k}$ *unconstrained design with the property that* $\mathrm{Min}\{l(h_1), l(h_2), \ldots, l(h_m)\}$ *is maximum among all the sets of generators. Under* (D, G) *every interaction is G-estimable.*

Denote by $R(n, k)$ the maximum resolution of the unconstrained 2^{n-k} designs. The G-estimability in (D, G) can be verified by observing that if $n \geqslant 2k$ and $k>3$, then $R(n, k) \geqslant 4$.

Indeed, denote by I_k the k-factor interaction $12\cdots k$. Let D be the 2^{n-k} design with subgroup of defining relations H generated by

$$\{1I_k(k+1), 2I_k(k+2), \ldots, kI_k(k+k)\}.$$

Then D has resolution 4 for $k>3$. Hence $R(2k, k) \geqslant 4$. Therefore $R(n, k) \geqslant 4$ if $n \geqslant 2k$ and $k>3$.

2.3.5. *The best* 2^{n-k} $(n-m>k>m)$ *fractional factorial design* (D, G) *is the design whose subgroup of defining relations is the same as the subgroup of defining relations of an unconstrained maximum resolution* $2^{(n-m)-k}$ *fractional factorial design on* $\{m+1, m+2, \ldots, n\}$. *Under* (D, G) *all the interactions on* $\{1, 2, \ldots, m\}$ *are G-estimable.*

2.3.6. *The best* 2^{n-k} $(k=n-m>m)$ *fractional factorial design* (D, G) *has subgroup of defining relations generated by* $\{12\cdots m(m+1),\ 12\cdots m(m+2), \ldots, 12\cdots mn\}$. *Under* (D, G) *all the interactions on* $\{1, 2, \ldots, m\}$ *are G-estimable, except the interaction* $12\cdots m$.

2.3.7. *For any* 2^{n-k} $(k>n-m \geqslant m)$ *fractional factorial design no interactions are G-estimable.*

This ends our exhaustive presentation of all cases that can occur in such a separation of factors into two disjoint groups. Demonstrations of some of the results are included in the appendix that follows. Detailed proofs are found in the doctoral dissertation of Xue (1994).

Acknowledgements

The authors thank David Sartori, manager at the PPG Research Center in Allison Park, Pennsylvania, for many motivational ideas and helpful conversations.

Appendix

Proof of 2.1.1. The subgroup of defining relations of the design D is $H=\{I, 12\cdots n\}$. Therefore, the aliasing patterns are $gH=\{g, g12\cdots n\}$. If g is not equal to 1 or $23\cdots n$, then either g or $g12\cdots n$ involves factor 1 and some other factors from the set $\{2,3,\ldots,n\}$. Hence, either g or $g12\cdots n$ is the zero interaction because of the nature of the bipartition. Therefore, g is either the zero interaction or the only interaction in a G-pattern. Hence, g is G-estimable. This completes the proof of the estimability.

Let D_0 be a 2^{n-1} fractional factorial design with defining relation $I=i_1i_2\cdots i_t$. The subgroup of defining relations of D_0 is $H_0=\{I, i_1i_2\cdots i_t\}$. If $1\notin\{i_1,i_2,\ldots,i_t\}$ then $ii_1i_2\cdots i_t$ is not zero for $i=2,3,\ldots,n$. We therefore have G-patterns $iH_G=\{i, ii_1i_2\cdots i_t\}$ for $i=2,3,\ldots,n$. Thus for $i=2,3,\ldots,n$ main effect i is not the only interaction in a G-pattern. Therefore, i is not estimable for $i=2,3,\ldots,n$. Hence, D is better than D_0. If $1\in\{i_1,i_2,\ldots,i_t\}$ and $t<n$, then without loss we may assume the defining relation to be $I=12\cdots t$. By arguments similar to those used in establishing the G-estimability under (D,G), every interaction is estimable under (D_0,G), except the main effect 1 and the interaction $23\cdots t$. Thus $m_{t-1}(D,G)<m_{t-1}(D_0,G)$. This completes the proof of 2.1.1.

Proof of 2.3.2. Estimability under (D,G) is easy to check. In addition, note that since $n-m\geqslant m>2$, the resolutions of H_1 and H_2 are $\geqslant 2$, hence all the main effects are G-estimable under (D,G).

Let D_0 be a 2^{n-2} fractional factorial design with subgroup of defining relations

$$H_0=\{I, A_1A_1', A_2A_2', A_3A_3'\},$$

where A_1,A_2,A_3, are interactions on $\{1,2,\ldots,m\}$ and A_1',A_2',A_3' are interactions on $\{m+1,m+2,\ldots,n\}$. If one of A_i's is empty, then the main effects $m+1,m+2,\ldots,n$ are not estimable. Hence, the design D is better than D_0. Similarly the design D is better than D_0 if one of A_1',A_2' and A_3' is empty. We can therefore assume that none of A_1,A_2,A_3,A_1',A_2' and A_3' be empty. Therefore, the non-estimable interactions in D_0 are A_1,A_2,A_3,A_1',A_2' and A_3'. Hence, D is better than D_0 unless $\{I,A_1,A_2,A_3\}$ and $\{I,A_1',A_2',A_3'\}$ are the subgroups of defining relations of a 2^{m-2} unconstrained minimum aberration fractional factorial design on $\{1,2,\ldots,m\}$ and a $2^{(n-m)-2}$ unconstrained minimum aberration design on $\{m+1,m+2,\ldots,n\}$, respectively. In such a case the estimability under (D_0,G) is as good as that under (D,G), but the resolution of D_0 is at most as big as that of D by the construction of D. This completes the proof of 2.3.2.

Proof of 2.3.5. Let D_0 be a 2^{n-k} fractional factorial design with subgroup of defining relations H_0. Since $k>m$ there exists at least one non-identity interaction h_0 in H_0 such that h_0 is an interaction on $\{m+1, m+2, \ldots, n\}$. Hence, no interactions on $\{m+1, m+2, \ldots, n\}$ are G-estimable, except h_0.

If there are at least two interactions in H_0 such that they only involve factors from $\{m+1, m+2, \ldots, n\}$ (this is always true when $k>m+1$), then no interactions on $\{m+1, m+2, \ldots, n\}$ are G-estimable. Therefore the only possibly G-estimable interactions are the interactions on $\{1, 2, \ldots, m\}$. If each interaction in H_0 only involves factors from $\{m+1, m+2, \ldots, n\}$, then the estimability under (D_0, G) is the same as that under (D, G). But D has maximum resolution. Hence D_0 is not better than D. We can therefore assume that H_0 contain some interaction h_1 which involves factors from $\{1, 2, \ldots, m\}$. If h_1 does not involve factors from $\{m+1, m+2, \ldots, n\}$, then no interactions on $\{1, 2, \ldots, m\}$ are G-estimable, except h_1. If h_1 involves factors from $\{m+1, m+2, \ldots, n\}$, then the projection of h_1 onto $\{m+1, m+2, \ldots, n\}$ is not G-estimable. In either case D is better than D_0.

If h_0 is the only non-identity interaction in H_0 which is an interaction on $\{m+1, m+2, \ldots, n\}$ (this case happens only when $k=m+1$), then the projection of H_0 on $\{1, 2, \ldots, m\}$ has kernel of order 2. Therefore, the range of the projection is the group that consists of all interactions on $\{1, 2, \ldots, m\}$. Hence, $1h^*$ is contained in H_0 for some interaction h^* on $\{m+1, m+2, \ldots, n\}$. Therefore, the main effect 1, which is aliased with h^*, is not G-estimable (note that h^* is not identity since we do not allow the subgroup of defining relations to contain main effects.). Hence the design D is better than D_0. This completes the proof of 2.3.5.

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