ON AZUMAYA AUTOMORPHISM EXTENSIONS OF RINGS

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ABSTRACT. Let B be a ring with 1, G an automorphism group of B of order n for some integer n invertible in B, C the center of B, and B^G the set of elements in B fixed under each element in G. Then, B is called an Azumaya automorphism extension of B^G with automorphism group G if $B \cong B^G \otimes_{C^G} V_B(B^G)$ as Azumaya C^G-algebras under the multiplication map. Some characterizations of an Azumaya automorphism extension are given and its subextensions arising from subgroups of G are also investigated.

1. INTRODUCTION

Let B be a ring with 1, G an automorphism group of B of order n for some integer n invertible in B, C the center of B, and B^G the set of elements in B fixed under each element in G. In [1], a class of Galois extensions called the Azumaya Galois extensions was studied as a generalization of the DeMeyer-Kanzaki Galois extensions ([2] and [6]) where B is called an Azumaya Galois extension with Galois group G if B is a Galois extension with Galois group G over an Azumaya C^G-algebra B^G and B a DeMeyer-Kanzaki Galois extension of B^G with Galois group G if B is an Azumaya algebra over C which is a Galois algebra with Galois group induced by and isomorphic with G ([2] and [6]). We note that an Azumaya Galois extension $B \cong B^G \otimes_{C^G} V_B(B^G)$ as C^G-algebras when $C \subset B^G$ where B^G is an Azumaya C^G-algebra and $V_B(B^G)$ is a central Galois algebra with Galois group induced by and isomorphic with G ([1], Theorem 1 and Theorem 2). Moreover,

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an Azumaya Galois extension B is characterized in terms of the Azumaya skew group ring $B * G$ over C^G ([1], Theorem 1). The purpose of the present paper is to study a class of rings B such that $B \cong B^G \otimes_{C^G} V_B(B^G)$ as Azumaya C^G-algebras under the multiplication map called an Azumaya automorphism extension of B^G with group G. Clearly, an Azumaya Galois extension B with $C \subset B^G$ is an Azumaya automorphism extension, but the converse is not true because $V_B(B^G)$ may not be a Galois algebra. We shall characterize an Azumaya automorphism extension in terms of the projective H-separable extension B over an Azumaya C^G-algebra B^G and the Azumaya C^G-algebra $\mathrm{Hom}_{B^G}(B, B)$ respectively. Moreover, we shall show some properties of the Azumaya automorphism extensions contained in B arising from subgroups of G. This work was done under the support of a Caterpillar Fellowship at Bradley University. The authors would like to thank Caterpillar Inc. for the support.

2. DEFINITIONS AND NOTATIONS

Throughout, B will represent a ring with 1, G an automorphism group of B of order n for some integer n invertible in B, C the center of B, and B^G the set of elements in B fixed under each element in G.

Let A be a subring of a ring B with the same identity 1. $V_B(A)$ is the commutator subring of A in B. We call B a separable extension of A if there exist $\{a_i, b_i$ in B, $i = 1, 2, ..., m$ for some integer $m\}$ such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where $\otimes$ is over A. An Azumaya algebra is a separable extension of its center. B is called a Galois extension of B^G with Galois group G if there exist elements $\{a_i, b_i$ in B, $i = 1, 2, ..., m\}$ for some integer m such that $\sum_{i=1}^{m} a_i g(b_i) = \delta_{1,g}$ for each $g \in G$. Such a set $\{a_i, b_i\}$ is called a G-Galois system for B. B is called an Azumaya Galois extension if B is a Galois extension with Galois group G over an Azumaya C^G-algebra B^G. We call B a DeMeyer-Kanzaki Galois extension of B^G with Galois group G if B is an Azumaya

C-algebra and C is a Galois algebra with Galois group $G|_C \cong G$. A ring B is called an H-separable extension of A if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B-bimodule, and B is called a Galois H-separable extension if it is a Galois and an H-separable extension of B^G (see [10]). B is called a central Galois extension if B is a Galois extension of C. We call B an Azumaya automorphism extension of B^G with group G if $B \cong B^G \otimes_{C^G} V_B(B^G)$ as Azumaya C^G-algebra under the multiplication map, and in particular, B is called a central Azumaya automorphism extension of B^G if $B^G = C$.

3. THE AZUMAYA AUTOMORPHISM EXTENSIONS

In this section, we shall characterize an Azumaya automorphism extension in terms of the projective H-separable extension B over an Azumaya C^G-algebra B^G and the Azumaya C^G-algebra $\mathrm{Hom}_{B^G}(B, B)$ respectively. We first give a lemma.

LEMMA 3.1.

Let B be a projective H-separable extension of B^G. Then $V_B(B^G)$ is a separable algebra over C and $C = C^G$.

PROOF. Since B is a projective H-separable extension of B^G, B is finitely generated projective B^G-module. Since $|G|^{-1} \in B$, B^G is a direct summand of B as B^G-bimodule, and so B is a B^G-progenerator. Hence $\mathrm{Hom}_{B^G}(B, B)$ is a separable extension of B ([9], Theorem 7-(3)). Therefore, $V_B(B^G)$ is separable over C ([9], Proposition 12-(1)). Moreover, since B is an H-separable extension of B^G and B^G is a direct summand of B, B^G satisfies the double centralizer property in B ([8], Proposition 1.2). Hence $C = V_B(B) \subset V_B(V_B(B^G)) = B^G$. Thus, $C = C^G$.

THEOREM 3.2.

The following statements are equivalent:

(1) B is an Azumaya automorphism extension of B^G with group G.

(2) B is a projective H-separable extension of B^G which is an Azumaya C^G-algebra.

(3) B is a projective H-separable extension of B^G and $\mathrm{Hom}_{B^G}(B,B)$ is an Azumaya C^G-algebra.

PROOF. (1) $\Longrightarrow$ (2) Since B is an Azumaya automorphism extension of B^G with group G, B^G is an Azumaya C^G-algebra. Hence we only need to prove that B is a projective H-separable extension of B^G. Since B is an Azumaya C^G-algebra, B is a projective C^G-module. Also, B^G is a separable C^G-algebra, so B is a projective B^G-module ([3], Proposition 2.3, page 48). But then B is an H-separable extension of B^G ([5], Lemma 1). Thus, B is a projective H-separable extension of B^G.

(2) $\Longrightarrow$ (1) Since B is a separable extension of B^G which is a separable C^G-algebra, B is a separable C^G-algebra by the transitivity property of separable extensions. Since B is a projective H-separable extension of B^G, $C = C^G$ by Lemma 3.1, and so B is an Azumaya C^G-algebra. But, by hypothesis, B^G is an Azumaya C^G-algebra, so $B \cong B^G \otimes_{C^G} V_B(B^G)$ as Azumaya C^G-algebras by the commutator theorem for Azumaya algebras ([3], Theorem 4.3, page 57).

(2) $\Longrightarrow$ (3) Since B is an Azumaya C^G-algebra, B is a C^G-progenerator ([3], Theorem 3.4, page 52). Therefore, $\mathrm{Hom}_{C^G}(B,B)$ is an Azumaya C^G-algebra ([3], Proposition 4.1, page 56). But, B^G is an Azumaya C^G-subalgebra of $\mathrm{Hom}_{C^G}(B,B)$, so $\mathrm{Hom}_{B^G}(B,B)$ $\big(= V_{\mathrm{Hom}_{C^G}(B,B)}(B^G)\big)$ is an Azumaya C^G-algebra by the commutator theorem for Azumaya algebras again.

(3) $\Longrightarrow$ (2) Since B is a projective H-separable extension of B^G, $V_B(B^G)$ is separable over C and $C = C^G$ by Lemma 3.1. Moreover, since $\mathrm{Hom}_{B^G}(B,B)$ is an Azumaya C^G-algebra and B is an H-separable extension of B^G, $\mathrm{Hom}_{B^G}(B,B) \cong B \otimes_{C^G} (V_B(B^G))^\circ$

([9], the proof of Proposition 12). Hence B and $(V_B(B^G))^\circ$ are Azumaya C^G-algebras ([3], Theorem 4.4, page 58). Hence $V_B(V_B(B^G))$ is an Azumaya C^G-algebra. But, by the proof of Lemma 3.1, $B^G = V_B(V_B(B^G))$, so B^G is an Azumaya C^G-algebra.

We note that if B is an Azumaya Galois extension with Galois group G and $C \subset B^G$, then $B \cong B^G \otimes_{C^G} V_B(B^G)$ as Azumaya C^G-algebras where $V_B(B^G)$ is a central Galois algebra with Galois group induced by and isomorphic with G ([1], Theorem 1 and Theorem 2). Hence the order of the Galois group $|G|$ is a unit in B ([6], Corollary 3).

4. THE AZUMAYA AUTOMORPHISM SUBEXTENSIONS

Let B be an Azumaya automorphism extension and a Galois extension of B^G with Galois group G. We shall show that any subgroup K of G induces an Azumaya automorphism subextension in B with group induced by K. Moreover, for any separable commutative subalgebra S of B, a sufficient condition is given for S such that $V_B(S)$ is an Azumaya automorphism subextension in B with group induced by a subgroup of G.

THEOREM 4.1.

Let B be an Azumaya automorphism extension and a Galois extension of B^G with Galois group G. Then, for any subgroup K of G, $B^K \cdot V_B(B^K)$ is an Azumaya automorphism extension of B^K with group K' induced by K.

PROOF. Since B is a Galois extension of B^G with Galois group G, B is also a Galois extension of B^K with Galois group K. Hence B is a finitely generated and projective left (or right) B^K-module. Moreover, since $|G|^{-1} \in B$, $|K|^{-1} \in B$. This implies that B^K is a direct summand of B as a B^K-module. Thus, the separability of B over C^G implies that B^K is a separable algebra over C^G by the proof of Theorem 3.8 on page 55 in [3]. But then $V_B(B^K)$ is also separable over C^G and $V_B(V_B(B^K)) = B^K$ by the commutator theorem for Azumaya algebras. Therefore, B^K and $V_B(B^K)$ have the same center which is denoted by D. This implies that B^K and $V_B(B^K)$ are Azumaya D-algebras, and so $B^K \otimes_D V_B(B^K) \cong B^K \cdot V_B(B^K)$ by the multiplication map. Noting that $B^K \cdot V_B(B^K)$ is invariant under K, we conclude that $B^K \cdot V_B(B^K)$ is an Azumaya automorphism extension of B^K with group K' induced by K.

By keeping the hypotheses of Theorem 4.1, next we give some equivalent conditions under which $B^K \cdot V_B(B^K)$ is a Galois extension of B^K with Galois group K' induced by K.

THEOREM 4.2.

Let B be an Azumaya automorphism extension and a Galois extension of B^G with Galois group G, and K a subgroup of G. Then, The following statements are equivalent:

(1) $B^K \cdot V_B(B^K)$ is a Galois extension of B^K with Galois group K' induced by K.

(2) $V_B(B^K)$ is a central Galois algebra with Galois group K' induced by K.

(3) $V_B(B^K) = \oplus \sum_{h' \in K'} J_{h'}$ where $J_{h'} = \{b \in V_B(B^K) \,\big|\, bx = h(x)b$ for all $x \in V_B(B^K)\}$.

(4) $B^I = B^K \cdot V_B(B^K)$ where $I = \{h \in K \,\big|\, h(d) = d$ for each $d \in V_B(B^K)\}$.

PROOF. (1) $\Longrightarrow$ (2) By Theorem 4.1, $B^K \cdot V_B(B^K)$ is an Azumaya automorphism extension of B^K with group K' induced by K. Moreover, by hypothesis, $B^K \cdot V_B(B^K)$ is a Galois extension of B^K with Galois group K' induced by K, so $B^K \cdot V_B(B^K)$ is an Azumaya Galois extension of B^K with Galois group K' induced by K. Hence $V_B(B^K)\big(= V_A(A^{K'})$ where $A = B^K \cdot V_B(B^K)\big)$ is a central Galois algebra with Galois group K' ([1], Theorem 2).

(2) $\Longrightarrow$ (1) Since $V_B(B^K)$ is a Galois extension with Galois group K' induced by K, $B^K \cdot V_B(B^K)$ is a Galois extension of B^K with the same Galois system.

(2) $\Longleftrightarrow$ (4) Since B is a Galois extension of B^G with Galois group G, B is also a Galois extension of B^K with Galois group K. Hence B is a finitely generated and projective left (or right) B^K-module. Moreover, B is an Azumaya C^G-algebra, so B is an H-separable extension of B^K ([5], Theorem 1). But then B is an H-separable Galois extension of B^K with Galois group K. Thus, (2) $\Longleftrightarrow$ (4) holds by Theorem 6-(3) in [10].

(2) $\Longrightarrow$ (3) Since $V_B(B^K)$ is a Galois algebra with Galois group K', $V_B(B^K) = \oplus \sum_{h' \in K'} J_{h'}$ by ([6], Theorem 1).

(3) $\Longrightarrow$ (2) Since $B^K \cdot V_B(B^K)$ is an Azumaya automorphism extension of B^K with group K', B^K and $V_B(B^K)$ are Azumaya algebras over the same center D. Hence K' is a D-automorphism group of $V_B(B^K)$. Therefore, $J_{h'} J_{h'^{-1}} = D$ for each $h' \in K'$ ([7], Lemma 5). Thus, $V_B(B^K)$ is a central Galois algebra with Galois group K' ([4], Theorem 1).

Let B be an Azumaya automorphism extension and a Galois extension of B^G with Galois group G. By Theorem 4.1, for any subgroup K of G, $B^K \cdot V_B(B^K)$ is an Azumaya automorphism extension of B^K with group K' induced by K. We shall give more properties of the Azumaya automorphism subextensions arising from subgroups of G. We first claim that $V_B(B^K)$ is a central Azumaya automorphism extension with group K'.

THEOREM 4.3.

Let B be an Azumaya automorphism extension and a Galois extension of B^G with Galois group G and K a nontrivial subgroup of G. Then, $V_B(B^K)$ is a central Azumaya automorphism extension with group K'.

PROOF. By Theorem 4.1, $V_B(B^K)$ is an Azumaya algebra over D with automorphism group K', so it suffices to show that $(V_B(B^K))^K = D$. In fact, by Theorem 4.1, B^K and

$V_B(B^K)$ are Azumaya algebras over the same center D. Hence $(V_B(B^K))^K = B^K \cap V_B(B^K) = D$.

Next, we show a one-to-one correspondence relation between a class of subgroups of G and a class of Azumaya automorphism subextensions in B. We begin with two lemmas.

LEMMA 4.4.

Let B be an Azumaya automorphism extension and a Galois extension of B^G with Galois group G, K a nontrivial subgroup of G, and D the center of B^K. Then, $V_B(D) = B^K \cdot V_B(B^K)$.

PROOF. By the proof of Theorem 4.1, B^K is a separable C^G-algebra, so D is a separable C^G-algebra ([3], Theorem 3.8, page 55). Therefore $V_B(D)$ is a separable C^G-algebra and $V_B(V_B(D)) = D$ by the commutator theorem for Azumaya algebras. This implies that $V_B(D)$ is an Azumaya D-algebra. But, by Theorem 4.1, $B^K \cdot V_B(B^K)$ is an Azumaya D-algebra, so $B^K \cdot V_B(B^K)$ is an Azumaya D-subalgebra of $V_B(D)$. Thus, $V_B(D) = (B^K \cdot V_B(B^K)) \cdot V_{V_B(D)}(B^K \cdot V_B(B^K))$ by the commutator theorem for Azumaya algebras again. Noting that $D \subset V_{V_B(D)}(B^K \cdot V_B(B^K)) \subset V_B(B^K \cdot V_B(B^K)) = V_{V_B(B^K)}(V_B(B^K)) = D$, we have $V_{V_B(D)}(B^K \cdot V_B(B^K)) = D$. Consequently, $V_B(D) = (B^K \cdot V_B(B^K)) \cdot D = B^K \cdot V_B(B^K)$.

LEMMA 4.5.

Assume that B is an Azumaya automorphism extension with group G. Let S be a commutative separable subalgebra of B over C^G and K a subgroup of G such that $S \subset B^K \subset V_B(S)$. If $V_B(S)$ is an Azumaya automorphism extension with group K' induced by K, then $V_B(S) = B^K \cdot V_B(B^K)$.

PROOF. We first note that $V_B(S)$ is invariant under K. Next, since $V_B(S)$ is an Azumaya automorphism extension with group K' induced by K, $V_B(S) = (V_B(S))^K \cdot V_{V_B(S)}((V_B(S))^K)$. Moreover, since $B^K \subset V_B(S)$, $(V_B(S))^K = B^K$. But then $V_{V_B(S)}((V_B(S))^K) = V_{V_B(S)}(B^K) = V_B(B^K) \cap V_B(S) = V_B(B^K)$ for $S \subset B^K$ by the definition of K. Thus, $V_B(S) = B^K \cdot V_B(B^K)$.

Let K and L be subgroups of G. We define $K \sim L$ if $B^L \cdot V_B(B^L) = B^K \cdot V_B(B^K)$. We note that $\sim$ is an equivalence relation on the class of subgroups of G, the equivalence class of K is denoted by $[K \sim]$, and $\mathcal{C} = \{[K \sim] \big| K < G\}$. Let $\mathcal{D} = \{A \big|$ there exists a commutative separable subalgebra S of B such that $A = V_B(S)$ is an Azumaya automorphism subextension in B with group K' induced by a subgroup K of G and $S \subset B^K \subset V_B(S)\}$.

THEOREM 4.6.

Assume that B is an Azumaya automorphism extension and a Galois extension of B^G with Galois group G. Let f: $\mathcal{C} \longrightarrow \mathcal{D}$ by $f([K \sim]) = B^K \cdot V_B(B^K)$ for each $[K \sim] \in \mathcal{C}$. Then f is a bijection.

PROOF. Clearly, f is well defined by Lemma 4.4, and an injection by the definition of $\sim$. Also, Lemma 4.5 implies that f is an surjection.

We conclude the present paper with two examples of Azumaya automorphism extensions with group G. One is a Galois extension with Galois group G and the other is not.

EXAMPLE 1.

Let $A = Q[i, j, k]$ be the quaternion algebra over the rational number Q, $B = M_2(A)$ the 2×2 matrix ring over A, and $G = \{1, g_i, g_j, g_k\}$ where

$g_i(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} iai^{-1} & ibi^{-1} \\ ici^{-1} & idi^{-1} \end{pmatrix}$, $g_j(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} jaj^{-1} & jbj^{-1} \\ jcj^{-1} & jdj^{-1} \end{pmatrix}$, and $g_k(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} kak^{-1} & kbk^{-1} \\ kck^{-1} & kdk^{-1} \end{pmatrix}$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B$. Then,

(1) The center of B is $C = \{\begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} |q \in Q\} \cong Q$, and $Q^G = Q$.

(2) $B^G = M_2(Q)$, the 2×2 matrix ring over Q. Hence B^G is an Azumaya Q-algebra.

(3) $V_B(B^G) = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} |a \in A\} \cong A$ which is a central Galois extension of Q with Galois group induced by and isomorphic with G with a Galois system: $\{\frac{1}{2},\ \frac{1}{2}i,\ \frac{1}{2}j,\ \frac{1}{2}k;\ \frac{1}{2},\ -\frac{1}{2}i,\ -\frac{1}{2}j,\ -\frac{1}{2}k\}$.

(4) $B \cong B^G \otimes_Q V_B(B^G)$ as Azumaya Q-algebras under the multiplication map.

(5) By (3), B is a Galois extension with Galois group G.

EXAMPLE 2.

Let B, A, and Q be as given in Example 1 and G the group generated by g and g_i where $g_i(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} iai^{-1} & ibi^{-1} \\ ici^{-1} & idi^{-1} \end{pmatrix}$ and $g(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} \alpha(a) & \alpha(b) \\ \alpha(c) & \alpha(d) \end{pmatrix}$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B$ where $\alpha(q_1 + q_2 i + q_3 j + q_4 k) = q_1 + q_2 j + q_3 k + q_4 i$ for all $q_1 + q_2 i + q_3 j + q_4 k \in A$. Then,

(1) It is straightforward to check that G has order 12.

(2) The center of B is $C = \{\begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} |q \in Q\} \cong Q$, and $Q^G = Q$.

(3) $B^G = M_2(Q)$, the 2×2 matrix ring over Q. Hence B^G is an Azumaya Q-algebra.

(4) $V_B(B^G) = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} |a \in A\} \cong A$ which is an Azumaya Q-algebra.

(5) $B \cong B^G \otimes_Q V_B(B^G)$ as Azumaya Q-algebras under the multiplication map.

(6) B is not a Galois extension with Galois group G. Suppose that B is a Galois extension with Galois group G. Then the skew group ring $B * G \cong \text{Hom}_{B^G}(B, B)$ ([2], Theorem 1). But B is a free module of rank 4 over B^G, so $B * G$ has rank 48 over B^G. On the other hand, $\text{Hom}_{B^G}(B, B)$ has rank 16 over B^G. This is a contradiction.

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