

# On a Class of Separable Hopf Galois Extensions

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ABSTRACT. Let  $H$  be a finite dimensional Hopf algebra over a field  $k$ ,  $H^*$  the dual Hopf algebra of  $H$ , and  $B$  a left  $H$ -module algebra. A separable  $H^*$ -Galois extension  $B$  is studied where the smash product  $B\#H$  is a Hirata separable extension of  $B$  and contains  $B$  as a direct summand as a  $B$ -bimodule. The class of  $B$  satisfying the above conditions is a generalization of the class of separable commutator  $H^*$ -Galois extensions.

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## 1 Introduction.

Let  $H$  be a finite dimensional Hopf algebra over a field  $k$ ,  $H^*$  the dual Hopf algebra of  $H$ ,  $B$  a left  $H$ -module algebra with center  $C$ , and  $B^H$  the  $H$ -invariant of  $B$ . Then  $B$  is called a (right)  $H^*$ -Galois extension of  $B^H$  if  $\beta : B \otimes_{B^H} B \longrightarrow B \otimes_k H^*$  is a bijection where  $\beta(a \otimes b) = (a \otimes 1)\rho(b)$  and  $\rho : B \longrightarrow B \otimes_k H^*$  is the structure map of  $B$ . The following classes of  $H^*$ -Galois extensions have been investigated ([4], [5], [9], [10]):

- (1) the  $H^*$ -Galois Azumaya extensions  $B$  of  $B^H$ , that is,  $B$  is a separable  $H^*$ -Galois extension of an Azumaya  $C^H$ -algebra  $B^H$  ([5]);
- (2) the DeMeyer-Kanzaki  $H^*$ -Galois extensions  $B$  of  $B^H$ , that is,  $B$  is an  $H^*$ -Galois Azumaya extension whose center  $C$  is an  $H^*$ -Galois extension of  $C^H$  ([9]);

(3) the separable commutator  $H^*$ -Galois extensions  $B$  of  $B^H$ , that is, the commutator of  $B^H$  in  $B$ ,  $V_B(B^H)$ , is a separable  $H^*$ -Galois extension of  $C^H$  ([4]); and

(4) the Hirata separable  $H^*$ -Galois extensions  $B$  of  $B^H$  ([10]).

We note that class (2)  $\subset$  class (1)  $\subset$  class (3). Let  $B$  be an  $H^*$ -Galois extension of  $B^H$ , it was shown that  $B$  is a Hirata separable of  $B^H$  (that is,  $B$  is in class (4)) if and only if  $V_B(B^H)$  is a left  $H$ -Galois extension of  $C$  ([10]). Thus, noting that  $B\#H$  is an  $H$ -Galois extension of  $B$ , we have that  $B\#H$  is a Hirata separable of  $B$  if and only if  $V_{B\#H}(B)$  is a left  $H^*$ -Galois extension of  $C^H$ . Moreover, when  $B\#H$  is a Hirata separable extension of  $B$ ,  $B$  is an  $H^*$ -Galois extension of  $B^H$ . Thus  $B$  is a separable  $H^*$ -Galois extension of  $B^H$  in case  $B\#H$  is a Hirata separable extension of  $B$  and contains  $B$  as a direct summand as a  $B$ -bimodule ([1]). The purpose of the present paper is to study the class of separable  $H^*$ -Galois extensions  $B$  where  $B\#H$  is a Hirata separable extension of  $B$  and contains  $B$  as a direct summand as a  $B$ -bimodule. We denote this class by class (5). We shall show that class (5) is a generalization of class (3) of separable commutator  $H^*$ -Galois extensions. A separable  $H^*$ -Galois extension in class (5) is characterized in terms of the separable left  $H^*$ -Galois extension  $V_{B\#H}(B)$ . Moreover, a sufficient condition is also given under which a  $B$  in class (5) becomes a separable commutator  $H^*$ -Galois extension in class (3) and an  $H^*$ -Galois Azumaya extension in class (1), respectively.

## 2 Basic Definitions and Notations.

Throughout,  $H$  denotes a finite dimensional Hopf algebra over a field  $k$  with comultiplication  $\Delta$  and counit  $\varepsilon$ ,  $H^*$  the dual Hopf algebra of  $H$ ,  $B$  a left  $H$ -module algebra,  $C$  the center of  $B$ ,  $B^H = \{b \in B \mid hb = \varepsilon(h)b \text{ for all } h \in H\}$  which is called the  $H$ -invariant of  $B$ , and  $B\#H$  the smash product of  $B$  and  $H$  where  $B\#H = B \otimes_k H$  such that for all  $b\#h$  and  $b'\#h'$  in  $B\#H$ ,  $(b\#h)(b'\#h') = \sum b(h_1 b')\#h_2 h'$  where  $\Delta(h) = \sum h_1 \otimes h_2$ .  $B$  is called a (right)  $H^*$ -Galois extension of  $B^H$  if  $B$  is a (right)  $H^*$ -comodule algebra with structure map  $\rho : B \rightarrow B \otimes_k H^*$  such that  $\beta : B \otimes_{B^H} B \rightarrow B \otimes_k H^*$  is a bijection where

$$\beta(a \otimes b) = (a \otimes 1)\rho(b).$$

For a subring  $A$  of  $B$  with the same identity  $1$ , we denote the commutator subring of  $A$  in  $B$  by  $V_B(A)$ . As given in [2], we call  $B$  a separable extension of  $A$  if there exist  $\{a_i, b_i$  in  $B, i = 1, 2, \dots, m$  for some integer  $m\}$  such that  $\sum a_i b_i = 1$ , and  $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$  for all  $b$  in  $B$  where  $\otimes$  is over  $A$ . An Azumaya algebra is a separable extension of its center. A ring  $B$  is called a Hirata separable extension of  $A$  if  $B \otimes_A B$  is isomorphic to a direct summand of a finite direct sum of  $B$  as a  $B$ -bimodule. An  $H^*$ -Galois extension  $B$  is called a commutator  $H^*$ -Galois extension if  $V_B(B^H)$  is an  $H^*$ -Galois extension of  $C^H$ , an  $H^*$ -Galois Azumaya extension if  $B$  is a separable  $H^*$ -Galois extension over  $B^H$  which is an Azumaya algebra over  $C^H$  ([5]), and a DeMeyer-Kanzaki  $H^*$ -Galois extension if  $B$  is an  $H^*$ -Galois Azumaya extension and  $V_B(B^H) = C$  ([9]). Throughout, an  $H^*$ -Galois extension means a right  $H^*$ -Galois extension unless it is stated otherwise.

### 3. Characterizations.

Let  $B$  be a separable  $H^*$ -Galois extension of  $B^H$  in class (5); that is,  $B\#H$  is a Hirata separable extension of  $B$  and contains  $B$  as a direct summand as a  $B$ -bimodule. We shall characterize  $B$  in terms of the separable left  $H^*$ -Galois extension  $V_{B\#H}(B)$ . We begin with some properties of such a  $B$ .

**Lemma 3.1.** *If  $B\#H$  is a Hirata separable extension of  $B$ , then (1) the center of  $B\#H$  is  $C^H$ , and (2)  $B$  is an  $H^*$ -Galois extension of  $B^H$ .*

*Proof.* (1) Since  $B\#H$  is a Hirata separable extension of  $B$  and a finitely generated and projective left  $B$ -module,  $V_{B\#H}(V_{B\#H}(B)) = B$  ([7], Theorem 1). Let  $Z$  be the center of  $B\#H$ . Then  $Z \subset C$ ; and so  $Z = C^H$  ([5], Lemma 3.2).

(2) Since  $B\#H$  is a Hirata separable extension of  $B$ ,  $B$  being a generator of itself is a generator of  $B\#H$  ([3], Lemma). Thus  $B$  is an  $H^*$ -Galois extension of  $B^H$  ([5], Theorem 2.2).

**Theorem 3.2.** *Let  $B$  be a left  $H$ -module algebra. Then  $B\#H$  is a Hirata separable extension of  $B$  and contains  $B$  as a direct summand as a  $B$ -bimodule (that is,  $B$  is in class (5)) if and only if  $V_{B\#H}(B)$  is a left  $H^*$ -Galois extension of  $C^H$  and  $B$  is a separable extension of  $B^H$ .*

*Proof.* ( $\implies$ ) Since  $B\#H$  is a Hirata separable extension of  $B$  and contains  $B$  as a direct summand as a  $B$ -bimodule,  $V_{B\#H}(B)$  is a separable algebra over the center of  $B\#H$  ([6], Proposition 1.3 and Proposition 1.4). By Lemma 3.1, the center of  $B\#H$  is  $C^H$ , so  $V_{B\#H}(B)$  is a separable  $C^G$ -algebra. Moreover, noting that  $B\#H$  is an  $H$ -Galois extension of  $B$  with the right  $H$ -comodule structure map  $B\#H \rightarrow (B\#H) \otimes_k H$  ([5], Example, page 45), we have that  $V_{B\#H}(B)$  is a left  $H^*$ -Galois extension of  $C^H$  where  $V_{B\#H}(B)$  is a right  $H$ -module algebra with the Ulbrich-Miyashita action ([10], Theorem 2.6). Next, we show that  $B$  is a separable extension of  $B^H$ . In fact, by Lemma 3.1,  $B$  is an  $H^*$ -Galois extension of  $B^H$ . Noting that  $B$  is also a Frobenius extension of  $B^H$  ([5], Corollary 5.2), we have that  $B$  is a separable extension of  $B^H$  because  $B$  is a direct summand of  $B\#H$  as a  $B$ -bimodule ([1], Theorem 1.8).

( $\impliedby$ ) Since  $V_{B\#H}(B)$  is a left  $H^*$ -Galois extension of  $C^H$ , the  $H$ -Galois extension  $B\#H$  is a Hirata separable of  $B$  ([10], Theorem 2.6); and so  $B$  is an  $H^*$ -Galois extension of  $B^H$  by Lemma 3.1. Also  $H$  is a finite dimensional Hopf algebra over a field  $k$ , so  $B$  is a Frobenius extension of  $B^H$  ([5], Corollary 5.2). By hypothesis,  $B$  is a separable extension of  $B^H$ , so  $B$  is a direct summand of  $B\#H$  as a  $B$ -bimodule ([1], Theorem 1.8).

We have an expression of  $B\#H$  for a separable  $H^*$ -Galois extension  $B$  as given by Theorem 3.2.

**Corollary 3.3.** *If  $B$  is a separable  $H^*$ -Galois extension as given by Theorem 3.2, then  $B\#H \cong B \otimes_C V_{B\#H}(B)$  as a left  $\Omega$ -module where  $\Omega = \text{Hom}_C(V_{B\#H}(B), V_{B\#H}(B))$ .*

*Proof.* Since  $B\#H$  is a Hirata separable extension of  $B$  and contains  $B$  as a direct

summand as a  $B$ -bimodule,  $V_{B\#H}(B)$  is a separable extension of  $C^H$  by the proof of Theorem 3.2, and  $V_{B\#H}(V_{B\#H}(B)) = B$  ([6], Proposition 1.3 and Proposition 1.4). Hence the center of  $V_{B\#H}(B)$  = the center of  $B = C$ . Therefore  $V_{B\#H}(B)$  is an Azumaya  $C$ -algebra. Let  $\Omega = \text{Hom}_C(V_{B\#H}(B), V_{B\#H}(B))$ . Then being a left  $\Omega$ -module,  $B\#H \cong V_{B\#H}(B) \otimes_C V_{B\#H}(V_{B\#H}(B))$ . Thus  $B\#H \cong V_{B\#H}(B) \otimes_C B$  ([2], Corollary 3.6, page 54) because  $V_{B\#H}(V_{B\#H}(B)) = B$ .

**Corollary 3.4.** *Let  $B$  be a separable  $H^*$ -Galois extension as given by Theorem 3.2. Then  $B\#H$  is a commutator  $H$ -Galois extension of  $B$  if and only if  $C = C^H$ .*

*Proof.* This is immediate by Corollary 3.3 and Lemma 2.8 in [4].

With the condition that  $C = C^H$ , we obtain more characterizations for a separable  $H^*$ -Galois extension  $B$  as given by Theorem 3.2.

**Theorem 3.5.** *Let  $B$  be a left  $H$ -module algebra such that  $C = C^H$ . Then the following statements are equivalent:*

- (1)  $B\#H$  is a Hirata separable extension of  $B$  and contains  $B$  as a direct summand as a  $B$ -bimodule, that is,  $B$  is in class (5);
- (2)  $V_{B\#H}(B)$  is an Azumaya  $C^H$ -algebra and  $B$  satisfies the double centralizer property in  $B\#H$ ;
- (3)  $V_{B\#H}(B)$  is a left  $H^*$ -Galois extension and an  $H$ -Galois extension of  $C^H$ .

*Proof.* (1)  $\implies$  (2) By the proof of Corollary 3.3,  $V_{B\#H}(B)$  is an Azumaya  $C$ -algebra and  $V_{B\#H}(V_{B\#H}(B)) = B$ . By hypothesis,  $C = C^H$ , so  $V_{B\#H}(B)$  is an Azumaya  $C^H$ -algebra and  $B$  satisfies the double centralizer property in  $B\#H$ .

(2)  $\implies$  (1) and (2)  $\implies$  (3) Since  $V_{B\#H}(B)$  is an Azumaya  $C^H$ -algebra and  $B\#H$  is a bimodule over  $V_{B\#H}(B)$ ,

$$\begin{aligned} B\#H &\cong V_{B\#H}(B) \otimes_{C^H} V_{B\#H}(V_{B\#H}(B)) \quad ([2], \text{Corollary 3.6, page 54}) \\ &\cong V_{B\#H}(B) \otimes_{C^H} B. \end{aligned}$$

Moreover, as given in the proof of Lemma 3.1, that  $V_{B\#H}(V_{B\#H}(B)) = B$  implies that the center of  $B\#H$  is  $C^H$ . Noting that  $B\#H \cong V_{B\#H}(B) \otimes_{C^H} B$  as a ring isomorphism, we have that the center of  $B$  is also  $C^H$ . Therefore  $V_{B\#H}(B)$  is an  $H$ -Galois extension of  $C^H$  from the fact that  $B\#H$  is an  $H$ -Galois extension of  $B$  ([4], Lemma 2.8). Furthermore, since  $B\#H \cong V_{B\#H}(B) \otimes_{C^H} B$  where  $V_{B\#H}(B)$  is an Azumaya  $C^G$ -algebra,  $B\#H$  is a Hirata separable extension of  $B$  and contains  $B$  as a direct summand as a  $B$ -bimodule. This proves (2)  $\implies$  (1). But then  $B\#H$  is a Hirata separable  $H$ -Galois extension of  $B$ . Therefore  $V_{B\#H}(B)$  is a left  $H^*$ -Galois extension of  $C^H$  ([9], Theorem 2.6). This proves (2)  $\implies$  (3).

(3)  $\implies$  (1) Since  $B\#H$  is an  $H$ -Galois extension of  $B$ , that  $V_{B\#H}(B)$  is a left  $H^*$ -Galois extension of  $B$  implies that  $B\#H$  is a Hirata separable of  $B$  ([9], Theorem 2.6). Moreover,  $V_{B\#H}(B)$  is an  $H$ -Galois extension of  $C^H$ , so  $B\#H \cong V_{B\#H}(B) \otimes_{C^H} B$  ([4], Lemma 2.8) such that  $C^H$  is a direct summand of  $V_{B\#H}(B)$  as a  $C^H$ -bimodule ([4], Corollary 1.10 and Proposition 1.9). Hence  $B$  is a direct summand of  $B\#H$  as a  $B$ -bimodule.

#### 4. Relations.

In this section, we shall show that the class of  $H^*$ -Galois Azumaya extensions and the class of separable commutator  $H^*$ -Galois extensions are contained in the class of separable  $H^*$ -Galois extensions  $B$  where  $B\#H$  is a Hirata separable extension of  $B$  and contains  $B$  as a direct summand as a  $B$ -bimodule. A sufficient condition is also given for the converse of each inclusion.

**Theorem 4.1.** *If  $B$  is a separable commutator  $H^*$ -Galois extension of  $B^H$ , then  $B\#H$  is a Hirata separable extension of  $B$  and contains  $B$  as a direct summand as a  $B$ -bimodule.*

*Proof.* Since  $B$  is a separable commutator  $H^*$ -Galois extension of  $B^H$ ,  $V_B(B^H)$  is

an  $H^*$ -Galois extension of  $C^H$  by definition. Hence  $B \cong B^H \otimes_{C^H} V_B(B^H)$  such that  $V_B(B^H)$  is a finitely generated and projective  $C^H$ -module ([4], Lemma 2.8). But  $C^H$  is commutative with 1, so  $V_B(B^H)$  is a progenerator of  $C^H$ . Thus  $\text{Hom}_{C^H}(V_B(B^H), V_B(B^H))$  is an Azumaya  $C^H$ -algebra ([2], Proposition 4.1, page 56). This implies that

$$\begin{aligned} B\#H &\cong (B^H \otimes_{C^H} V_B(B^H))\#H \cong B^H \otimes_{C^H} (V_B(B^H)\#H) \\ &\cong B^H \otimes_{C^H} \text{Hom}_{C^H}(V_B(B^H), V_B(B^H)), \end{aligned}$$

and so  $B\#H$  is a Hirata separable extension of  $B^H$  and contains  $B^H$  as a direct summand as a  $B^H$ -bimodule because  $\text{Hom}_{C^H}(V_B(B^H), V_B(B^H))$  is an Azumaya  $C^H$ -algebra. Furthermore, noting that  $V_B(B^H)$  is a separable  $C^H$ -algebra, we have that  $\text{Hom}_{C^H}(V_B(B^H), V_B(B^H))$  is a finitely generated and projective left module over  $V_B(B^H)$ . But then  $\text{Hom}_{C^H}(V_B(B^H), V_B(B^H))$  is a Hirata separable extension of  $V_B(B^H)$  ([3], Theorem 1). Thus  $B^H \otimes_{C^H} \text{Hom}_{C^H}(V_B(B^H), V_B(B^H))$  is a Hirata separable extension of  $B^H \otimes_{C^H} V_B(B^H)$ ; and so  $B\#H$  is a Hirata separable extension of  $B$ . Also, by hypothesis,  $V_B(B^H)$  is a separable extension of  $C^H$ , so  $B$  is a separable extension of  $B^H$ . Therefore  $B$  is a direct summand of  $B\#H$  as a  $B$ -bimodule ([1], Theorem 1.8).

For a Hirata separable extension  $B$  of  $B^H$ , the converse of the Theorem 4.1 holds.

**Theorem 4.2.** *Let  $B$  be a Hirata separable extension of  $B^H$ . If  $B\#H$  is a Hirata separable extension of  $B$  and contains  $B$  as a direct summand as a  $B$ -bimodule, then  $B$  is a separable commutator  $H^*$ -Galois extension of  $B^H$ .*

*Proof.* Since  $B\#H$  is a Hirata separable extension of  $B$ ,  $B$  is an  $H^*$ -Galois extension of  $B^H$  by Lemma 3.1. Thus  $B\#H \cong \text{Hom}_{B^H}(B, B)$  ([5], Theorem 2.2). By hypothesis,  $\text{Hom}_{B^H}(B, B)$  ( $\cong B\#H$ ) and  $B$  are Hirata separable extensions of  $B$  and  $B^H$  respectively, so  $V_B(B^H)$  is an Azumaya  $C^H$ -algebra ([8], Proposition 12). Therefore  $B \cong V_B(B^H) \otimes_{C^H} V_B(V_B(B^H))$  ([2], Corollary 3.6, page 54). But  $B$  is a Hirata separable and  $H^*$ -Galois extension of  $B^H$ , so  $V_B(V_B(B^H)) = B$  ([10], Lemma 2.5). Then we

have that  $B \cong V_B(B^H) \otimes_{C^H} B^H$  where  $V_B(B^H)$  is a finitely generated and projective  $C^H$ -module. Consequently,  $V_B(B^H)$  be an  $H^*$ -Galois extension of  $C^H$  ([4], Lemma 2.8); and so  $B$  is a separable commutator  $H^*$ -Galois extension of  $B^H$ .

Next is a relationship between the class of  $H^*$ -Galois Azumaya extensions and the class of separable  $H^*$ -Galois extensions  $B$  where  $B\#H$  is a Hirata separable extension of  $B$  and contains  $B$  as a direct summand as a  $B$ -bimodule.

**Theorem 4.3.** *If  $B$  is an  $H^*$ -Galois Azumaya extension of  $B^H$ , then  $B\#H$  is a Hirata separable extension of  $B$  and contains  $B$  as a direct summand as a  $B$ -bimodule.*

*Proof.* Since  $B$  is an  $H^*$ -Galois Azumaya extension of  $B^H$ ,  $B\#H$  is an Azumaya  $C^H$ -algebra ([5], Theorem 3.4). Noting that  $B\#H$  is a finitely generated and projective left module over  $B$ , we have that  $B\#H$  is a Hirata separable extension of  $B$  ([3], Theorem 1). But  $B$  is a separable Frobenius extension of  $B^H$ , so  $B$  is a direct summand of  $B\#H$  as a  $B$ -bimodule ([1], Theorem 1.8).

We now give a sufficient condition under which the converse of Theorem 4.3 holds.

**Theorem 4.4.** *Let  $B$  be a separable  $H^*$ -Galois extension of  $B^H$  where  $B\#H$  is a Hirata separable extension of  $B$  and contains  $B$  as a direct summand as a  $B$ -bimodule. If  $B\#H$  is a separable  $C^H$ -algebra, then  $B$  is an  $H^*$ -Galois Azumaya extension of  $B^H$ .*

*Proof.* Since  $B\#H$  is a Hirata separable extension of  $B$  and contains  $B$  as a direct summand as a  $B$ -bimodule, the center of  $B\#H$  is  $C^H$  by Lemma 3.1. By hypothesis,  $B\#H$  is a separable  $C^H$ -algebra, so it is an Azumaya  $C^H$ -algebra. Since  $B$  is a direct summand of  $B\#H$  as a  $B$ -bimodule,  $B$  is a separable  $C^H$ -algebra by the proof of Theorem 3.8 on page 55 in [2]. Thus  $B$  is an  $H^*$ -Galois Azumaya extension of  $B^H$  ([5], Theorem 3.4).

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