

Skew Group Rings which are Galois

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Abstract. Let $S * G$ be a skew group ring of a finite group G over a ring S . It is shown that if $S * G$ is an G' -Galois extension of $(S * G)^{G'}$ where G' is the inner automorphism group of $S * G$ induced by the elements in G , then S is an G -Galois extension of S^G . A necessary and sufficient condition is also given for the commutator subring of $(S * G)^{G'}$ in $S * G$ to be a Galois extension, where $(S * G)^{G'}$ is the subring of the elements fixed under each element in G' .

Key Words and Phrases: Skew group rings, Azumaya algebras, Galois extensions, H -separable extensions.

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1. INTRODUCTION

Let S be a ring with 1, C the center of S , G a finite automorphism group of S of order n invertible in S , S^G the subring of the elements fixed under each element in G , $S * G$ a skew group ring of group G over S , and G' the inner automorphism group of $S * G$ induced by the elements in G , that is, $g'(x) = gxg^{-1}$ for each g in G and x in $S * G$, so the restriction of G' to S is G . In [1] and [2], an G -Galois extension S of S^G which is an Azumaya C^G -algebra is characterized in terms of the Azumaya C^G -algebra $S * G$ and the H -separable extension $S * G$ of S respectively, and properties of the commutator subring of S in $S * G$ are given in [3]. It is clear that S is an G -Galois extension of S^G implies that $S * G$ is an G' -Galois extension of $(S * G)^{G'}$ with the same Galois system as S . In the present paper, we shall show the converse theorem: if $S * G$ is an G' -Galois extension of $(S * G)^{G'}$, then S is an G -Galois extension of S^G . Moreover, for an G' -Galois extension $S * G$ of $(S * G)^{G'}$ which is a projective separable C^G -algebra, S can be shown to

be an G -Galois extension of S^G which is also a projective separable C^G -algebra. Then a sufficient condition on $(S * G)^{G'}$ is given for S to be an G -Galois extension of S^G which is an Azumaya C^G -algebra, and an equivalent condition on S^G is obtained for the commutator subring of $(S * G)^{G'}$ in $S * G$ to be an G -Galois extension. This work was supported by a Caterpillar Fellowship at Bradley University. We would like to thank Caterpillar Inc. for the support.

2. PRELIMINARIES

Throughout, we keep the notations as given in Introduction. Let B be a subring of a ring A with 1. Following [1] and [2], A is called a separable extension of B if there exist $\{a_i, b_i$ in A , $i = 1, 2, \dots, m$ for some integer $m\}$ such that $\sum a_i b_i = 1$, and $\sum s a_i \otimes b_i = \sum a_i \otimes b_i s$ for all s in A where \otimes is over B . An Azumaya algebra is a separable extension of its center. A ring A is called an H -separable extension of B if $A \otimes_B A$ is isomorphic to a direct summand of a finite direct sum of A as an A -bimodule. It is known that an Azumaya algebra is an H -separable extension and an H -separable extension is a separable extension. Let S be given as in section 1. Then it is called an G -Galois extension of S^G if there exist elements $\{c_i, d_i$ in S , $i = 1, 2, \dots, k$ for some integer $k\}$ such that $\sum c_i g_j(d_i) = \delta_{1,j}$, where $G = \{g_1, g_2, \dots, g_n\}$ with identity g_1 , for each $g_j \in G$. Such a set $\{c_i, d_i\}$ is called an G -Galois system for S .

3. GALOIS SKEW GROUP RINGS

In this section, we shall show that an G' -Galois extension skew group ring $S * G$ implies an G -Galois extension S . More results are obtained for S^G when $(S * G)^{G'}$ is a projective separable C^G -algebra, and an H -separable S^G -extension respectively.

THEOREM 3.1. If $S * G$ is an G' -Galois extension of $(S * G)^{G'}$, then S is an G -Galois extension of S^G .

PROOF. Let $\{u_i, v_i | i = 1, 2, \dots, m\}$ be a G' -Galois system of $S * G$ over $(S * G)^{G'}$, that is, u_i and v_i are elements of $S * G$ satisfying $\sum_{i=1}^m u_i g'(v_i) = \sum_{i=1}^m u_i g v_i g^{-1} = \delta_{1,g}$. Let $w_i = \sum_{h \in G} h v_i$, $i = 1, 2, \dots, m$. Then $g w_i = \sum_{h \in G} g h v_i = w_i$. Since $\{h | h \in G\}$ is a basis of $S * G$ over S , we have $u_i = \sum_{h \in G} s_h^{(u_i)} h$ and $w_i = \sum_{h \in G} s_h^{(w_i)} h$, $i = 1, 2, \dots, m$, for some $s_h^{(u_i)}, s_h^{(w_i)}$ in S . Let $x_i = \sum_{h \in G} s_h^{(u_i)}$ and $y_i = s_1^{(w_i)}$, $i = 1, 2, \dots, m$. We will prove that $\{x_i, y_i | i = 1, 2, \dots, m\}$ is a G -Galois system for S over S^G . First we prove that

- (1) $g(s_h^{(w_i)}) = s_{gh}^{(w_i)}$ for all $i = 1, 2, \dots, m$ and all $g, h \in G$, and
- (2) $\sum_{i=1}^m u_i w_i = 1$.

For (1), since $w_i = gw_i$, we have

$$\sum_{k \in G} s_k^{(w_i)} k = \sum_{h \in G} s_h^{(w_i)} h = g \sum_{h \in G} s_h^{(w_i)} h = \sum_{h \in G} g s_h^{(w_i)} h = \sum_{h \in G} g(s_h^{(w_i)})gh$$

Since $\{k | k \in G\}$ is a basis of $S * G$ over S , $g(s_h^{(w_i)}) = s_{gh}^{(w_i)}$.

For (2), since $\{u_i, v_i | i = 1, 2, \dots, m\}$ is a G' -Galois system for $S * G$ over $(S * G)^{G'}$, $\sum_{i=1}^m u_i h'(v_i) = \sum_{i=1}^m u_i h v_i h^{-1} = \delta_{1,h}$. Therefore,

$$1 = \sum_{h \in G} \delta_{1,h} h = \sum_{h \in G} \left(\sum_{i=1}^m u_i h v_i h^{-1} \right) h = \sum_{h \in G} \sum_{i=1}^m u_i h v_i = \sum_{i=1}^m u_i \sum_{h \in G} h v_i = \sum_{i=1}^m u_i w_i$$

Next we prove that $\{x_i, y_i | i = 1, 2, \dots, m\}$ is a G -Galois system for S over S^G . By using (1) and (2), we have

$$\begin{aligned} 1 &= \sum_{i=1}^m u_i w_i = \sum_{i=1}^m \left(\sum_{h \in G} s_h^{(u_i)} h \right) \left(\sum_{k \in G} s_k^{(w_i)} k \right) \\ &= \sum_{i=1}^m \sum_{h \in G} \sum_{k \in G} s_h^{(u_i)} h s_k^{(w_i)} k = \sum_{i=1}^m \sum_{h \in G} \sum_{k \in G} s_h^{(u_i)} h (s_k^{(w_i)}) h k \\ &= \sum_{i=1}^m \sum_{g \in G} \sum_{hk=g} s_h^{(u_i)} h (s_k^{(w_i)}) h k = \sum_{i=1}^m \sum_{g \in G} \sum_{hk=g} s_h^{(u_i)} s_{hk}^{(w_i)} h k \quad \text{by (1)} \\ &= \sum_{i=1}^m \sum_{g \in G} \sum_{h \in G} s_h^{(u_i)} s_{hh^{-1}g}^{(w_i)} h h^{-1} g \quad (\text{since } hk = g, k = h^{-1}g) \\ &= \sum_{i=1}^m \sum_{g \in G} \sum_{h \in G} s_h^{(u_i)} s_g^{(w_i)} g \\ &= \sum_{g \in G} \left(\sum_{i=1}^m \sum_{h \in G} s_h^{(u_i)} s_g^{(w_i)} \right) g. \end{aligned}$$

Hence $\sum_{i=1}^m \sum_{h \in G} s_h^{(u_i)} s_g^{(w_i)} = \delta_{1,g}$. But $x_i = \sum_{h \in G} s_h^{(u_i)}$, $y_i = s_1^{(w_i)}$, and $g(s_1^{(w_i)}) = s_g^{(w_i)}$ by (1), so

$$\sum_{i=1}^m x_i g(y_i) = \sum_{i=1}^m \sum_{h \in G} s_h^{(u_i)} g(s_1^{(w_i)}) = \sum_{i=1}^m \sum_{h \in G} s_h^{(u_i)} s_g^{(w_i)} = \delta_{1,g}.$$

We shall show more properties of the G -Galois extension S of S^G when $S * G$ is an G' -Galois extension of $(S * G)^{G'}$ which possesses a property.

THEOREM 3.2. If $S * G$ is an G' -Galois extension of $(S * G)^{G'}$ which is a projective separable C^G -algebra, then S is an G -Galois extension of S^G which is also a projective separable C^G -algebra.

PROOF. Since $S * G$ is an G' -Galois extension of $(S * G)^{G'}$, S is an G -Galois extension of S^G by theorem 3.1. Since $S * G$ is an G' -Galois extension of $(S * G)^{G'}$ again, it is a separable extension [4]. Also $(S * G)^{G'}$ is a separable C^G -algebra, so $S * G$ is separable C^G -algebra by the transitivity of separable extensions. Next, we claim that S is also a separable C^G -algebra. In fact, since n is a unit in S , the trace map: $\frac{1}{n}(tr_G(\)) : S \longrightarrow S^G \longrightarrow 0$ is a splitting homomorphism of the imbedding homomorphism of S^G into S as a two sided S^G -module. Hence S^G is a direct summand of S . Since S is a direct summand of $S * G$ as an S^G -bimodule, S^G is so of $S * G$ as an S^G -module. Moreover, S is a finite generated and projective S^G -module (for S is an G -Galois extension of S^G), so $S * G$ is a finite generated and projective S^G -module by the transitivity of finite generated and projective modules. This implies that S^G is a projective separable C^G -algebra by the proof of Lemma 2 ([4], p.120).

THEOREM 3.3. If (i) $S * G$ is an G' -Galois extension of $(S * G)^{G'}$ and (ii) $(S * G)^{G'}$ is an H -separable extension of S^G which is a separable C^G -algebra, then S is an G -Galois extension of S^G which is an Azumaya C^G -algebra.

PROOF. Since $S * G$ is an G' -Galois extension of $(S * G)^{G'}$ with an inner Galois group G' , $S * G$ is an H -separable extension of $(S * G)^{G'}$ ([5], Proposition 4). By hypothesis, $(S * G)^{G'}$ is an H -separable extension of S^G , so $S * G$ is an H -separable extension of S^G by the transitivity of H -separable extensions. Noting that n is an unit of S , we have that S^G is an S^G -direct summand of S . But S is a direct summand of $S * G$ as an S^G -module, so S^G is a direct summand of $S * G$ as an S^G -module. Thus $V_{S * G}(V_{S * G}(S^G)) = S^G$ ([6], Proposition 1.2). This implies that the center of $S * G$ is contained in S^G , and so the center of $S * G$ is C^G . Therefore, $S * G$ is an Azumaya C^G -algebra. Thus S^G is an Azumaya C^G -algebra. Consequently S is an G -Galois extension of S^G which is an Azumaya C^G -algebra ([2], Theorem 3.1).

4. GALOIS COMMUTATOR SUBRINGS.

In [5], the class of G -Galois and H -separable extension was studied. Let A be an G -Galois and H -separable extension of A^G and the commutator subring of A^G in A be $V_A(A^G)$. Then, $V_A(A^G)$ is a central (G/I) -Galois algebra if and only if $A^I = A^G(V_A(A^G))$, where $I = \{g \in G \mid g(d) = d \text{ for all } d \in V_A(A^G)\}$ ([5], Theorem 6.3). Applying such

an equivalence condition to an G' -Galois extension $S * G$, we shall characterize a Galois commutator subring $V_{S * G}((S * G)^{G'})$ in terms of elements in S^G .

In the following, we denote the center of G by P and the center $S * G$ by Z . By a direct computation, we have:

LEMMA 4.1.

- (1) Let $I = \{g_i \in G \mid g'_i(d) = d \text{ for each } d \in ZG\}$. Then $I = P$.
- (2) Let x be an element in $(S * G)^{G'}$. Then $x = \sum_{i=1}^n s_i g_i$ such that $g_j(s_i) = s_k$ whenever $g_j g_i g_j^{-1} = g'_j(g_i) = g_k \in G$.

LEMMA 4.2. Assume that $S * G$ is an G' -Galois extension of $(S * G)^{G'}$ and an Azumaya Z -algebra. Then $V_{S * G}((S * G)^{G'})$ is a central (G'/P') -Galois algebra if and only if $S^P G = (S * G)^{G'} G$.

PROOF. Since n is a unit in Z and $S * G$ is an Azumaya Z -algebra, $V_{S * G}((S * G)^{G'}) = V_{S * G}(V_{S * G}(ZG)) = ZG$ by the commutator theorem for Azumaya algebras ([7], Theorem 4.3) (for ZG is a separable Z -subalgebra). Moreover, since $S * G$ is an G' -Galois extension of $(S * G)^{G'}$ with an inner Galois group G' , it is an H -separable extension of $(S * G)^{G'}$ ([5], Proposition 4). Hence $V_{S * G}((S * G)^{G'}) (= ZG)$ is a central (G'/P') -Galois algebra if and only if $(S * G)^{P'} = (S * G)^{G'} ZG$ by Lemma 4.1-(1) and Theorem 6.3 in [5]. Clearly, $Z \subset (S * G)^{G'}$, so $(S * G)^{G'} ZG = (S * G)^{G'} G$. Noting that P is the center of G , we have $(S * G)^{P'} = S^P G$. Thus the Lemma holds.

THEOREM 4.4. Assume that $S * G$ is an G' -Galois extension of $(S * G)^{G'}$ and an Azumaya Z -algebra. Then ZG is a central (G'/P') -Galois algebra if and only if for every $s \in S^P$ there exists an $n \times n$ matrix $[s_{k,h}]_{k,h \in G}$ for some $s_{k,h}$ in S such that (1) $\sum_{h \in G} s_{gh^{-1},h} = \delta_{1,g} s$ (therefore $s = \sum_{h \in G} s_{h^{-1},h}$), and (2) $g(s_{k,h}) = s_{gkg^{-1},h}$ for every $g \in G$.

PROOF. \implies : Assume that ZG is a central (G'/P') -Galois algebra. Then by Lemma 4.2, $S^P G = (S * G)^{G'} G$. Therefore, for every $s \in S^P$, $s = s1 \in S^P G = (S * G)^{G'} G$. Hence there exists $\sum_{k \in G} s_{k,h} k \in (S * G)^{G'}$ for each $h \in G$ such that

$$s = s1 = \sum_{h \in G} \left(\sum_{k \in G} s_{k,h} k \right) h = \sum_{g \in G} \left(\sum_{kh=g} s_{k,h} \right) g = \sum_{g \in G} \left(\sum_{h \in G} s_{gh^{-1},h} \right) g.$$

Since $\{g \mid g \in G\}$ is a basis of $S * G$ over S , we have $\sum_{h \in G} s_{gh^{-1},h} = \delta_{1,g} s$, and therefore $\sum_{h \in G} s_{h^{-1},h} = s$. Furthermore, for each $h \in G$, $\sum_{k \in G} s_{k,h} k \in (S * G)^{G'}$, i.e., $\sum_{k \in G} s_{k,h} k$

$= g \sum_{k \in G} s_{k,h} k g^{-1} = \sum_{k \in G} g(s_{k,h}) g k g^{-1}$ for every $g \in G$. Therefore $g(s_{k,h}) = s_{gkg^{-1},h}$ for every $g \in G$ since $\{k \mid k \in G\}$ is a basis of $S * G$ over S .

\Leftarrow : Assume for every $s \in S^P$ there exists an $n \times n$ matrix $[s_{k,h}]_{k,h \in G}$ such that $\sum_{h \in G} s_{gh^{-1},h} = \delta_{1,g} s$ and $g(s_{k,h}) = s_{gkg^{-1},h}$ for every $g \in G$. Then

$$g\left(\sum_{k \in G} s_{k,h} k\right)g^{-1} = \sum_{k \in G} g(s_{k,h}) g k g^{-1} = \sum_{k \in G} s_{gkg^{-1},h} g k g^{-1} = \sum_{k \in G} s_{k,h} k,$$

that is, $\sum_{k \in G} s_{k,h} k \in (S * G)^{G'}$ for every $h \in G$. Therefore

$$s = \sum_{g \in G} \delta_{1,g} s g = \sum_{g \in G} \left(\sum_{h \in G} s_{gh^{-1},h}\right) g = \sum_{g \in G} \left(\sum_{k h = g} s_{k,h}\right) g = \sum_{h \in G} \left(\sum_{k \in G} s_{k,h} k\right) h \in (S * G)^{G'} G.$$

Hence, for every $s \in S^P$ and every $g \in G$, $sg \in (S * G)^{G'} G G = (S * G)^{G'} G$, that is, $S^P G \subseteq (S * G)^{G'} G$.

On the other hand, for any $\sum_{k \in G} s_k k \in (S * G)^{G'}$, we have $\sum_{k \in G} s_k k = g \sum_{k \in G} s_k k g^{-1}$ ■
 $= \sum_{k \in G} g(s_k) g k g^{-1}$ for every $g \in G$. Therefore $g(s_k) = s_{gkg^{-1}}$ for every $g \in G$ since $\{k \mid k \in G\}$ is a basis of $S * G$ over S . In particular, for every $p \in P$, $p(s_k) = s_{p k p^{-1}} = s_k$, i.e., $s_k \in S^P$ for every $k \in G$, and therefore $\sum_{k \in G} s_k k \in S^P G$ if $\sum_{k \in G} s_k k \in (S * G)^{G'}$. Hence $(S * G)^{G'} \subseteq S^P G$. Therefore, $(S * G)^{G'} G \subseteq S^P G G = S^P G$. Hence $S^P G = (S * G)^{G'} G$. So $(S * G)^{P'} = S^P G = (S * G)^{G'} G = (S * G)^{G'} ZG$. Consequently, by Lemma 4.2, ZG is a central (G'/P') -Galois algebra.

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