

On Hopf Galois Hirata Extensions

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ABSTRACT. Let H be a finite dimensional Hopf algebra over a field k , H^* the dual Hopf algebra of H , and B a right H^* -Galois and Hirata separable extension of B^H . Then B is characterized in terms of the commutator subring $V_B(B^H)$ of B^H in B and the smash product $V_B(B^H)\#H$. A sufficient condition is also given for B to be an H^* -Galois Azumaya extension of B^H .

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1. Introduction. Let H be a finite dimensional Hopf algebra over a field k , H^* the dual Hopf algebra of H , and B a right H^* -Galois extension of B^H . In [3], the class of H^* -Galois Azumaya extensions were investigated, and in [8], it was shown that B is a Hirata separable extension of B^H if and only if the commutator subring $V_B(B^H)$ of B^H in B is a left H -Galois extension of C where C is the center of B ([8], Lemma 2.1 and Theorem 2.6). The purpose of the present paper is to characterize a right H^* -Galois and Hirata separable extension B of B^H in terms of the commutator subring $V_B(B^H)$ and the smash product $V_B(B^H)\#H$. Let B be a right H^* -Galois extension of B^H . Then the following statements are equivalent:

- (1) B is a Hirata separable extension of B^H ,
- (2) $V_B(B^H)$ is an Azumaya C -algebra and $V_B(V_B(B^H)) = B^H$,
- (3) $V_B(B^H)$ is a right H^* -Galois extension of C and a direct summand of $V_B(B^H)\#H$ as a $V_B(B^H)$ -bimodule, and

(4) $V_B(B^H)$ is a right H^* -Galois extension of C , and $V_B(B^H)\#H$ is a direct summand of a finite direct sum of $V_B(B^H)$ as a bimodule over $V_B(B^H)$.

Moreover, an equivalent condition is given for a right H^* -Galois and Hirata separable extension B of B^H to be an H^* -Galois Azumaya extension which was studied in [3] and [7]. Also, let B be a right H^* -Galois and Hirata separable extension of B^H and A a subalgebra of B^H over C such that B^H is a projective Hirata separable extension of A containing A as a direct summand as an A -bimodule. Then $V_{B^H}(A)$ is a separable subalgebra of B^H over C , and there exists an H -submodule algebra D in B which is separable over C such that $D^H = V_{B^H}(A)$ and $D \cong V_{B^H}(A) \otimes_Z F$ as Azumaya Z -algebras where Z is the center of D and F is an Azumaya Z -algebra in D .

2. Basic definitions and notations. Throughout, H denotes a finite dimensional Hopf algebra over a field k with comultiplication Δ and counit ε , H^* the dual Hopf algebra of H , B a left H -module algebra, C the center of B , $B^H = \{b \in B \mid hb = \varepsilon(h)b \text{ for all } h \in H\}$ which is called the H -invariant of B , and $B\#H$ the smash product of B with H where $B\#H = B \otimes_k H$ such that for all $b\#h$ and $b'\#h'$ in $B\#H$, $(b\#h)(b'\#h') = \sum b(h_1b')\#h_2h'$ where $\Delta(h) = \sum h_1 \otimes h_2$. B is called a right H^* -Galois extension of B^H if B is a right H^* -comodule algebra with structure map $\rho : B \rightarrow B \otimes_k H^*$ such that $\beta : B \otimes_{B^H} B \rightarrow B \otimes_k H^*$ is a bijection where $\beta(a \otimes b) = (a \otimes 1)\rho(b)$.

For a subring A of B with the same identity 1, we denote the commutator subring of A in B by $V_B(A)$. we call B a separable extension of A if there exist $\{a_i, b_i \text{ in } B, i = 1, 2, \dots, m \text{ for some integer } m\}$ such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where \otimes is over A . An Azumaya algebra is a separable extension of its center. A ring B is called a Hirata separable extension of A if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B -bimodule. A right H^* -Galois extension B is called an H^* -Galois Azumaya extension if B is separable over B^H which is an Azumaya algebra over C^H . A right H^* -Galois extension B of B^H is called an H^* -Galois Hirata extension if B is also a Hirata separable extension of B^H . Throughout, an H^* -Galois

extension means a right H^* -Galois extension unless it is stated otherwise.

3. The H^* -Galois Hirata extensions. In this section, we shall characterize an H^* -Galois Hirata extension B of B^H in terms of the commutator subring $V_B(B^H)$ of B^H in B and the smash product $V_B(B^H)\#H$. A relationship between an H^* -Galois Hirata extension and an H^* -Galois Azumaya extension is also given. We begin with some properties of an H^* -Galois Hirata extension B of B^H .

LEMMA 3.1. *If A_1 and A_2 are H^* -Galois extensions such that $A_1^H = A_2^H$ and $A_1 \subset A_2$, then $A_1 = A_2$.*

PROOF. By Theorem 5.1 in [3], there exist $\{x_i, y_i \in A_1 \mid i = 1, 2, \dots, n\}$ for some integer n such that, for all $h \in H$, $\sum x_i(hy_i) = T(h)1_{A_1}$ where $T \in \int_{H^*}^r$, the set of right integrals in H^* . Let $t \in \int_H^l$, the set of left integrals in H , such that $T(t) = 1$, then $\{x_i, f_i = t(y_i-)\mid i = 1, 2, \dots, n\}$ is a dual basis of the finitely generated and projective right module over A_1^H . Since $A_1 \subset A_2$ such that $A_1^H = A_2^H$, $\{x_i, f_i \mid i = 1, 2, \dots, n\}$ is also a dual basis of the finitely generated and projective right module A_2 over A_1^H . This implies that $A_1 = A_2$.

LEMMA 3.2. *If B is an H^* -Galois Hirata extension of B^H , then B^H is a direct summand of B as a B^H -bimodule.*

PROOF. We use the argument as given in [2]. Since B is an H^* -Galois and a Hirata separable extension of B^H , $V_B(B^H)$ is a left H -Galois extension of C ([8], Lemma 2.1 and Theorem 2.6). Hence $V_B(B^H)$ is a finitely generated and projective module over C ([3], Theorem 2.2). Let $\Omega = \text{Hom}_C(V_B(B^H), V_B(B^H))$. Since C is commutative, $V_B(B^H)$ is a progenerator of C . Thus B is a left Ω -module such that $B \cong V_B(B^H) \otimes_C \text{Hom}_\Omega(V_B(B^H), B) \cong V_B(B^H) \otimes_C V_B(V_B(B^H))$ as C -algebras where $f(1) \in B^H$ for each $f \in \text{Hom}_\Omega(V_B(B^H), B)$ by the proof of Lemma 2.8 in [2]. But $V_B(V_B(B^H)) = B^H$ ([2], Lemma 2.5), so $B \cong V_B(B^H) \otimes_C B^H$. This implies that $V_B(B^H)$ is an H^* -Galois extension of C ([2], Lemma 2.8); and so C is a direct summand of $V_B(B^H)$ as a C -bimodule

([2], Corollary 1.9 and Corollary 1.10). Therefore B^H is a direct summand of B as a B^H -bimodule.

By the proof of Lemma 3.2, $V_B(B^H)$ is an H^* -Galois extension of C .

COROLLARY 3.3. *If B is an H^* -Galois Hirata extension of B^H , then $V_B(B^H)$ is an H^* -Galois extension of C .*

COROLLARY 3.4. *If B is an H^* -Galois Hirata extension of B^H , then $B = B^H \cdot V_B(B^H)$ and the centers of B , B^H , and $V_B(B^H)$ are the same C .*

PROOF. By Corollary 3.3, $V_B(B^H)$ is an H^* -Galois extension of C , so $B^H \cdot V_B(B^H)$ is also an H^* -Galois extension of B^H ($= (B^H \cdot V_B(B^H))^H$) with the same Galois system as $V_B(B^H)$ ([3], Theorem 5.1). Noting that $B^H \cdot V_B(B^H) \subset B$, we conclude that $B = B^H \cdot V_B(B^H)$ by Lemma 3.1. Moreover, $V_B(V_B(B^H)) = B^H$ ([8], Lemma 2.5), so the centers of B^H , $V_B(B^H)$, and B are the same C .

THEOREM 3.5. *Let B be an H^* -Galois extension of B^H . The following statements are equivalent:*

- (1) B is a Hirata separable extension of B^H ,
- (2) $V_B(B^H)$ is an H^* -Galois extension of C and a direct summand of $V_B(B^H) \# H$ as a $V_B(B^H)$ -bimodule,
- (3) $V_B(B^H)$ is an Azumaya C -algebra and $V_B(V_B(B^H)) = B^H$, and
- (4) $V_B(B^H)$ is an H^* -Galois extension of C and $V_B(B^H) \# H$ is a direct summand of a finite direct sum of $V_B(B^H)$ as a bimodule over $V_B(B^H)$.

PROOF. (1) \implies (3) Since B is an H^* -Galois and a Hirata separable extension of B^H , by Lemma 3.2, B^H is a direct summand of B as a B^H -bimodule. Thus $V_B(V_B(B^H)) = B^H$ and $V_B(B^H)$ is a separable C -algebra ([4], Proposition 1.3 and Proposition 1.4). But the center of $V_B(B^H)$ is C by Corollary 3.4, so $V_B(B^H)$ is an Azumaya C -algebra.

(3) \implies (1) Since $V_B(B^H)$ is an Azumaya C -algebra and B is a bimodule over $V_B(B^H)$, $B \cong V_B(B^H) \otimes_C V_B(V_B(B^H)) = V_B(B^H) \otimes_C B^H$ as a bimodule over $V_B(B^H)$ ([1], Corollary

3.6 on page 56). Noting that $B \cong V_B(B^H) \otimes_C B^H$ is also an isomorphism as C -algebras and that $V_B(B^H)$ is an Azumaya C -algebra, we conclude that $V_B(B^H) \otimes_C B^H$ is a Hirata separable extension of B^H ; and so B is a Hirata separable extension of B^H .

(3) \implies (2) By the proof of (3) \implies (1) $B \cong V_B(B^H) \otimes_C B^H$ such that $V_B(B^H)$ is a finitely generated and projective module over C , so $V_B(B^H)$ is an H^* -Galois extension of C ([2], Lemma 2.8). Moreover, since $V_B(B^H)$ is an Azumaya C -algebra, $V_B(B^H)$ is a direct summand of $V_B(B^H) \otimes_C (V_B(B^H))^\circ$ as a $V_B(B^H)$ -bimodule where $(V_B(B^H))^\circ$ is the opposite algebra of $V_B(B^H)$. But $V_B(B^H) \otimes_C (V_B(B^H))^\circ \cong \text{Hom}_C(V_B(B^H), V_B(B^H)) \cong V_B(B^H) \# H$ ([3], Theorem 2.2), so $V_B(B^H)$ is a direct summand of $V_B(B^H) \# H$ as a $V_B(B^H)$ -bimodule.

(2) \implies (3) Since $V_B(B^H)$ is an H^* -Galois extension of C , $B^H \cdot V_B(B^H)$ is an H^* -Galois extension of $(B^H \cdot V_B(B^H))^H$. But $(B^H \cdot V_B(B^H))^H = B^H$, so $B^H \cdot V_B(B^H)$ and B are H^* -Galois extensions of B^H such that $B^H \cdot V_B(B^H) \subset B$. Hence $B^H \cdot V_B(B^H) = B$ by Lemma 3.1. Thus the centers of B and $V_B(B^H)$ are the same C . Moreover, $V_B(B^H)$ is a direct summand of $V_B(B^H) \# H$ as a $V_B(B^H)$ -bimodule by hypothesis, so it is separable C -algebra ([3], Theorem 2.3). Thus $V_B(B^H)$ is an Azumaya C -algebra. But then $B \cong V_B(B^H) \otimes_C V_B(V_B(B^H))$. On the other hand, by hypothesis, $V_B(B^H)$ is an H^* -Galois extension of C , so $B \cong V_B(B^H) \otimes_C B^H$ ([2], Lemma 2.8). Therefore $V_B(V_B(B^H)) = B^H$.

(3) \iff (4) Since $V_B(B^H)$ is an H^* -Galois extension of C , it is a finitely generated and projective module over C and $\text{Hom}_C(V_B(B^H), V_B(B^H)) \cong V_B(B^H) \# H$ ([3], Theorem 2.2). But then $V_B(B^H)$ is a Hirata separable extensions of C if and only if $V_B(B^H) \# H$ is a direct summand of a finite direct sum of $V_B(B^H)$ as a bimodule over $V_B(B^H)$ ([6], Corollary 3). Thus $V_B(B^H)$ is an Azumaya C -algebra if and only if $V_B(B^H)$ is an H^* -Galois extension of C and $V_B(B^H) \# H$ is a direct summand of a finite direct sum of $V_B(B^H)$ as a bimodule over $V_B(B^H)$.

By Theorem 3.5, we can obtain a relationship between the class of H^* -Galois Hirata extensions and the class of H^* -Galois Azumaya extensions which were studied in [3] and

[7].

COROLLARY 3.6. *Let B be an H^* -Galois Azumaya extension of B^H . Then B is an H^* -Galois Hirata extension of B^H if and only if $C = C^H$.*

PROOF. (\implies) Since B is an H^* -Galois Hirata extension of B^H , $V_B(B^H)$ is an Azumaya algebra over C and a left H -Galois extension of C ([8], Theorem 2.6). Hence $V_B(V_B(B^H)) = B^H$ ([8], Lemma 2.5). Thus $C \subset B^H$; and so $C = C^H$.

(\impliedby) Since B is an H^* -Galois Azumaya extension of B^H , $V_B(B^H)$ is separable over C^H ([3], Lemma 4.1). Since B is an H^* -Galois Azumaya extension of B^H again, $V_B(B^H)$ is an H^* -Galois extension of $(V_B(B^H))^H$ ([3], Lemma 4.1), so both $B^H \cdot V_B(B^H)$ and B are H^* -Galois extensions of B^H such that $B^H \cdot V_B(B^H) \subset B$. Hence $B^H \cdot V_B(B^H) = B$ by Lemma 3.1. This implies that the center of $V_B(B^H)$ is C . But by hypothesis, $C = C^H$, so $V_B(B^H)$ is an Azumaya C -algebra. Hence $V_B(B^H)$ is a Hirata separable extension of C . But $B = B^H \cdot V_B(B^H) \cong B^H \otimes_C V_B(B^H)$ as Azumaya C -algebras, so B is a Hirata separable extension of B^H . Thus B is an H^* -Galois Hirata extension of B^H .

COROLLARY 3.7. *Let B be an H^* -Galois Hirata extension of B^H . Then B is an H^* -Galois Azumaya extension of B^H if and only if B is an Azumaya C^H -algebra.*

PROOF. (\implies) Since B is an H^* -Galois Azumaya extension of B^H , B^H is Azumaya C^H -algebra and B is separable over B^H ([3], Theorem 3.4). Hence B is separable over C^H by the transitivity of separable extensions. But B is an H^* -Galois Azumaya extension of B^H and an H^* -Galois Hirata extension of B^H by hypothesis, so $C = C^H$ by Corollary 3.6. This implies that B is an Azumaya C^H -algebra.

(\impliedby) By hypothesis, B is an Azumaya C^H -algebra. Hence $C = C^H$. But B is an H^* -Galois Hirata extension of B^H , so $V_B(B^H)$ is an Azumaya subalgebra of B over C by Theorem 3.5-(3). Since B is an H^* -Galois Hirata extension of B^H again, B is a Hirata separable extension of B^H and a finitely generated and projective module over B^H . Thus $V_B(V_B(B^H)) = B^H$ ([5], Theorem 1); and so $B^H (= V_B(V_B(B^H)))$ is an Azumaya

subalgebra of B over C^H by the commutator theorem for Azumaya algebras ([1], Theorem 4.3 on page 57). This proves that B is an H^* -Galois Azumaya extension of B^H .

4. Invariant subalgebras. For an H^* -Galois Hirata extension B , let A be a subalgebra of B^H over C such that B^H is a projective Hirata separable extension of A and contains A as a direct summand as an A -bimodule. In this section, we shall show that $V_{B^H}(A)$ is the H -invariant subalgebra of a separable subalgebra D in B over C , that is, $D^H = V_{B^H}(A)$. We denote by \mathcal{S} the set $\{A \mid A \text{ is a subalgebra of } B^H \text{ over } C \text{ such that } B^H \text{ is a projective Hirata separable extension of } A \text{ and contains } A \text{ as a direct summand as an } A\text{-bimodule}\}$.

LEMMA 4.1. *Let B be an H^* -Galois Hirata extension of B^H . For any $A \in \mathcal{S}$, $V_B(A)$ is an H -submodule algebra of B and separable over C , and $(V_B(A))^H = V_{B^H}(A)$ which is a separable C -algebra.*

PROOF. Since $A \in \mathcal{S}$, B^H is a projective Hirata separable extension of A and contains A as a direct summand as an A -bimodule. But B is an H^* -Galois Hirata extension of B^H , so B is a projective Hirata separable extension of B^H . Hence, by the transitivity property of projective Hirata separable extensions, B is a projective Hirata separable extension of A . Also B^H is a direct summand of B as a B^H -bimodule by Lemma 3.2, so A is a direct summand of B as an A -bimodule. Thus $V_B(A)$ is a separable algebra over C ([5], Theorem 1). Moreover, it is clear that $(V_B(A))^H = V_{B^H}(A) = V_{B^H}(A)$, so $V_{B^H}(A)$ is a separable C -algebra (Corollary 3.4 and [5], Theorem 1).

Next we want to show which separable subalgebra of B^H over C is an H -invariant subring of an H -submodule algebra in B . Let $\mathcal{T} = \{E \subset B \mid E \text{ is a separable } C\text{-subalgebra of } B^H \text{ and satisfies the double centralizer property in } B^H \text{ such that } V_{B^H}(E) \in \mathcal{S}\}$. Next we show that for any $E \in \mathcal{T}$, E is the H -invariant subring of an H -submodule algebra D in B which is separable over C .

THEOREM 4.2. *Let E be in \mathcal{T} . Then there exists an H -submodule algebra D in B which is separable over C such that $D^H = E$.*

PROOF. Since E is in \mathcal{T} , $V_{B^H}(E)$ is in \mathcal{S} such that $V_{B^H}(V_{B^H}(E)) = E$. Now by Lemma 4.1, $V_B(V_{B^H}(E))$ is an H -submodule algebra of B and separable over C such that $(V_B(V_{B^H}(E)))^H = V_{B^H}(V_{B^H}(E))$. But $V_{B^H}(V_{B^H}(E)) = E$, so $(V_B(V_{B^H}(E)))^H = E$. Let $D = V_B(V_{B^H}(E))$. Then D satisfies the theorem.

By Theorem 4.2, we obtain an expression for the separable H -submodule algebra D for a given E in \mathcal{T} .

COROLLARY 4.3. *By keeping the notations as given in Theorem 4.2, let Z be the center of E . Then $D \cong E \otimes_Z V_D(E)$ as Azumaya Z -algebras.*

PROOF. Since E satisfies the double centralizer property in B^H , $V_{B^H}(V_{B^H}(E)) = E$. Hence the centers of E and $V_{B^H}(E)$ are the same Z . Similarly as given in the proof of Lemma 4.1, since $V_{B^H}(E)$ is in \mathcal{S} , $B (= B^H \cdot V_B(B^H))$ is a projective Hirata separable extension of $V_{B^H}(E)$ and contains $V_{B^H}(E)$ as a direct summand as a $V_{B^H}(E)$ -bimodule by the transitivity property of projective Hirata separable extensions and the direct summand conditions. Thus $V_{B^H}(E)$ satisfies the double centralizer property in B , that is, $V_B(V_B(V_{B^H}(E))) = V_{B^H}(E)$. This implies that the centers of $V_{B^H}(E)$ and $V_B(V_{B^H}(E))$ are the same. Therefore D and E have the same center Z . Noting that D and E are separable C algebras by Theorem 4.2, we conclude that $E (= D^H)$ is an Azumaya subalgebra of D over Z ; and so $D \cong E \otimes_Z V_D(E)$ as Azumaya Z -algebras ([1], Theorem 4.3 on page 57).

Remark. When B is an H^* -Galois Azumaya extension of B^H , the correspondence $A \rightarrow V_B(A)$ as given in Lemma 4.1 recovers the one-to-one correspondence between the set of separable subalgebras of B^H and the set of H^* -Galois extensions in B containing $V_B(B^H)$ as given in [3].

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