

ON HOPF DEMEYER-KANZAKI GALOIS EXTENSIONS

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ABSTRACT. Let H be a finite dimensional Hopf algebra over a field k , B a left H -module algebra, and H^* the dual Hopf algebra of H . For an H^* -Azumaya Galois extension B with center C , it is shown that B is an H^* -DeMeyer-Kanzaki Galois extension if and only if C is a maximal commutative separable subalgebra of the smash product $B\#H$. Moreover, the characterization of a commutative Galois algebra as given by S. Ikehata is generalized.

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1 Introduction. Let H be a finite dimensional Hopf algebra over a field k , B a left H -module algebra, and H^* the dual Hopf algebra of H . In [7], the class of Azumaya Galois extensions of a ring as studied in [1] and [2] was generalized to H^* -Azumaya Galois extensions. An H^* -Azumaya Galois extension B was characterized in terms of the smash product $B\#H$ ([7], Theorem 3.4). Observing that the commutator $V_B(B^H)$ of B^H in B is also an H^* -Azumaya Galois extension ([7], Lemma 4.1), in the present paper, we shall give a characterization of an H^* -Azumaya Galois extension B in terms of $V_B(B^H)$. Moreover, we shall investigate the class of H^* -Azumaya Galois extensions B such that $V_B(B^H) = C$ where C is the center of B . We note that when $H = kG$ where G is a finite automorphism group of B , such a B is precisely a DeMeyer-Kanzaki Galois extension with Galois group G ([3],[6],[8],[9]). Several equivalent conditions are then given for an H^* -Azumaya Galois extension being an H^* -DeMeyer-Kanzaki Galois extension, and the characterization of a commutative Galois algebra as given by S. Ikehata ([5], Theorem 2) is generalized to an H^* -DeMeyer-Kanzaki Galois extension.

2 Basic Definitions and Notations. Throughout, H denotes a finite dimensional Hopf algebra over a field k with comultiplication Δ and counit ε , H^* the dual Hopf algebra of H , B a left H -module algebra, C the center of B , $B^H = \{b \in B \mid hb = \varepsilon(h)b \text{ for all } h \in H\}$, and $B\#H$ the smash product of B with H where $B\#H = B \otimes_k H$ such that for all $b\#h$ and $b'\#h'$ in $B\#H$, $(b\#h)(b'\#h') = \sum b(h_1b')\#h_2h'$ where $\Delta(h) = \sum h_1 \otimes h_2$.

For a subring A of B with the same identity 1, we denote the commutator subring of A in B by $V_B(A)$. We call B a separable extension of A if there exist $\{a_i, b_i$ in B , $i = 1, 2, \dots, m$ for some integer $m\}$ such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where \otimes is over A . An Azumaya algebra is a separable extension of its center. A ring B is called a Hirata separable extension of A if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B -bimodule. B is called an H^* -Galois extension of B^H if B is a right H^* -comodule algebra with structure map $\rho : B \rightarrow B \otimes_k H^*$ such that $\beta : B \otimes_{B^H} B \rightarrow B \otimes_k H^*$ is a bijection where $\beta(a \otimes b) = (a \otimes 1)\rho(b)$. An H^* -Galois extension B is called an H^* -Azumaya Galois extension if B is separable over B^G which is an Azumaya algebra over C^G , and an H^* -DeMeyer-Kanzaki Galois extension if B is an H^* -Azumaya Galois extension and $V_B(B^H) = C$.

Let P be a finitely generated and projective module over a commutative ring R . Then for a prime ideal p of R , $P_p (= P \otimes_R R_p)$ is a free module over $R_p (= \text{the local ring of } R \text{ at } p)$, and the rank of P_p over R_p is the number of copies of R_p in P_p , that is, $\text{rank}_{R_p}(P_p) = m$ for some integer m . It is known that the $\text{rank}_R(P)$ is a continuous function ($\text{rank}_R(P)(p) = \text{rank}_{R_p}(P_p) = m$) from $\text{Spec}(R)$ to the set of nonnegative integers with the discrete topology (see [4, Corollary 4.11, page 31]). We shall use the $\text{rank}_R(P)$ -function for a finitely generated and projective module P over a commutative ring R .

3. H^* -Azumaya Galois extensions. In this section, keeping all notations as given in section 2, we shall characterize an H^* -Azumaya Galois extension B in terms of the commutator $V_B(B^H)$ of B^H in B .

THEOREM 3.1. *If $B = B^H \cdot V_B(B^H)$, then $(V_B(B^H))^H = C^H$.*

PROOF. Since $C \subset V_B(B^H)$, $C^H \subset (V_B(B^H))^H$. Conversely, since $V_B(B^H) \subset B$, $(V_B(B^H))^H \subset B^H$. Hence $(V_B(B^H))^H \subset B^H \cap V_B(B^H) \subset$ the center of $V_B(B^H)$. But $B = B^H \cdot V_B(B^H)$, so the center of $V_B(B^H)$ is C . Thus, $(V_B(B^H))^H \subset C^H$.

THEOREM 3.2. *B is an H^* -Azumaya Galois extension of B^H if and only if $B = B^H \cdot V_B(B^H)$ such that $V_B(B^H)$ is an H^* -Azumaya Galois extension of C^H and B^H is an Azumaya C^H -algebra.*

PROOF. (\implies) Since B is an H^* -Azumaya Galois extension of B^H , $V_B(B^H)$ is an H^* -Azumaya Galois extension of $(V_B(B^H))^H$ ([7], Lemma 4.1) and B^H is an Azumaya C^H -algebra ([7], Theorem 3.4). Moreover, by the proof of Lemma 4.1 in [7], $B\#H$ is an Azumaya C^H -algebra such that $B\#H \cong B^H \otimes_{C^H} (V_B(B^H)\#H) \cong B^H(V_B(B^H)\#H)$ where B^H and $V_B(B^H)\#H$ are Azumaya C^H -algebras. But H is a finite dimensional Hopf algebra over a field k , so $B \cong B^H \otimes_{C^H} V_B(B^H)$ from the isomorphism $B\#H \cong B^H \otimes_{C^H} (V_B(B^H)\#H)$; and so $B = B^H \cdot V_B(B^H)$. Hence $(V_B(B^H))^H = C^H$ by Lemma 3.1. Thus $V_B(B^H)$ is an H^* -Azumaya Galois C^H -algebra.

(\impliedby) Since $V_B(B^H)$ is an H^* -Azumaya Galois algebra over C^H , $V_B(B^H)\#H$ is an Azumaya C^H -algebra ([7], Theorem 3.4). By hypothesis, B^H is an Azumaya C^H -algebra, so $B^H \otimes_{C^H} (V_B(B^H)\#H) \cong B^H V_B(B^H)\#H = B\#H$ which is an Azumaya C^H -algebra. Thus $B\#H$ is a Hirata separable extension of B ([5], Theorem 1). Moreover, $V_B(B^H)$ is a separable C^H -algebra ([7], Theorem 3.4) and B^H is an Azumaya C^H -algebra by hypothesis, so $B^H \cdot V_B(B^H)$ ($= B$) is also a separable C^H -algebra. Thus B is an H^* -Azumaya Galois extension of B^H ([7], Theorem 3.4).

Next we generalize the characterization of a commutative Galois algebra as given by S. Ikehata ([5], Theorem 2) to a commutative H^* -Galois algebra.

LEMMA 3.3. *If C is a commutative H^* -Galois algebra over C^H , then C is a maximal commutative subalgebra of $C\#H$.*

PROOF. Since C is a commutative H^* -Galois algebra over C^H , $C\#H \cong \text{Hom}_{C^H}(C, C)$ ([6], Theorem 1.7). Hence it suffices to show that $V_{\text{Hom}_{C^H}(C, C)}(C_L) = C_L$ where $C_L = \{c_L, \text{ the left multiplication map induced by } c \in C\}$. In fact, $C_L \subseteq V_{\text{Hom}_{C^H}(C, C)}(C_L)$ is clear. Conversely, Let $f \in V_{\text{Hom}_{C^H}(C, C)}(C_L)$. Then for each $c \in C$, $(cf)(x) = (fc)(x)$ for all $x \in C$. Hence $cf(x) = f(cx)$; and so $cf(1) = f(c)$ for all $c \in C$. Thus $f(c) = d_f(c)$ for all $c \in C$ where $d_f = f(1) \in C$, that is, $f = (d_f)_L \in C_L$.

THEOREM 3.4. *Let C be a commutative separable C^H -algebra containing C^H as a direct summand as a C^H -module. Then, C is a commutative H^* -Galois algebra over C^H if and only if $C \otimes_{C^H} (C\#H) \cong M_n(C)$, the matrix algebra over C of order n where n is the dimension of H over k .*

PROOF. (\implies) Since C is an H^* -Galois algebra over C^H , $C\#H \cong \text{Hom}_{C^H}(C, C)$ such that C is finitely generated and projective over C^H ([6], Theorem 1.7). Hence $C\#H$ is an Azumaya C^H -algebra and C is a maximal commutative subalgebra of the Azumaya C^H -algebra $C\#H$ by Lemma 3.3. By hypothesis, C is also a separable C^H -algebra, so C is a splitting ring for the Azumaya C^H -algebra $C\#H$ such that $C \otimes_{C^H} (C\#H) \cong \text{Hom}_C(C\#H, C\#H)$ (see the proof of Theorem 5.5 on page 64 in [4]). Noting that $C\#H = C \otimes_k H$ which is a free C -module of rank n where $n = \dim_k(H)$, we have that $C \otimes_{C^H} (C\#H) \cong M_n(C)$.

(\impliedby) Since $C \otimes_{C^H} (C\#H) \cong M_n(C)$, $C \otimes_{C^H} (C\#H)$ is an Azumaya C -algebra. By hypothesis, C^H is a direct summand of C as a C^H -module, so $C\#H$ is an Azumaya C^H -algebra ([4], Corollary 1.10, page 45). Hence $C\#H$ is a Hirata separable extension of C . But C is a separable C^H -algebra by hypothesis, so C is an H^* -Galois algebra over C^H ([7], Theorem 3.4).

We remark that the necessity does not need the hypothesis that C^H is a direct summand of C .

4. H^* -DeMeyer-Kanzaki Galois extensions. We recall that B is an H^* -DeMeyer-Kanzaki Galois extension of B^H if B is an H^* -Azumaya Galois extension of B^H and $V_B(B^H) = C$. In this section, we characterize an H^* -DeMeyer-Kanzaki Galois extension in terms of the smash product $V_B(B^H)\#H$ and prove that C is a splitting ring for the Azumaya C^H -algebras $V_B(B^H)\#H$ and $B\#H$.

THEOREM 4.1. *Let B be an H^* -Azumaya Galois extension of B^H . Then the following statements are equivalent:*

- (1) B is an H^* -DeMeyer-Kanzaki Galois extension of B^H .
- (2) $\text{rank}_{C^H}(V_B(B^H)) = \text{rank}_{C^H}(C)$.
- (3) C is a maximal commutative separable subalgebra of $V_B(B^H)\#H$.

PROOF. (1) \implies (2) It is clear.

(2) \implies (1) Since B is an H^* -Azumaya Galois extension of B^H , $V_B(B^H)$ is an H^* -Azumaya Galois algebra over C^H by Theorem 3.2 such that $V_B(B^H)$ is a separable and finitely generated projective module over C^H ([7], theorem 3.4). Hence the rank function $\text{rank}_{C^H}(V_B(B^H))$ is defined, and $V_B(B^H)$ is an Azumaya algebra over its center ([4], Theorem 3.8, page 55). But $B = B^H \cdot V_B(B^H)$ by Theorem 3.2, so the center of $V_B(B^H)$ is C . Thus $V_B(B^H)$ is an Azumaya C -algebra; and so C is a direct summand $V_B(B^H)$ as a C -module. This implies that C is a direct summand $V_B(B^H)$ as a C^H -module. Therefore the rank function $\text{rank}_{C^H}(C)$ is also defined. Now by hypothesis, $\text{rank}_{C^H}(V_B(B^H)) = \text{rank}_{C^H}(C)$, so $V_B(B^H) = C$, that is, B is an H^* -DeMeyer-Kanzaki Galois extension of B^H .

(1) \implies (3) Since B is an H^* -DeMeyer-Kanzaki Galois extension of B^H , B is an H^* -Azumaya Galois extension such that $V_B(B^H) = C$. Hence $B = B^H \cdot V_B(B^H) \cong B^H \otimes_{C^H} C$ such that C is an H^* -Galois algebra over C^H by Theorem 3.2; and so C is a separable C^H -algebra containing C^H as a direct summand as a C^H -module ([7], theorem 3.4). Hence C is a maximal commutative separable subalgebra of $C \# H$ where $C = V_B(B^H)$ by Lemma 3.3.

(3) \implies (2) Since B is an H^* -Azumaya Galois extension of B^H , $B = B^H \cdot V_B(B^H) \cong B^H \otimes_{C^H} V_B(B^H)$ such that $V_B(B^H)$ is an H^* -Azumaya Galois algebra over C^H by Theorem 3.2. Hence $V_B(B^H) \# H$ is an Azumaya C^H -algebra and $V_B(B^H)$ is an Azumaya C -algebra ([7], Theorem 3.4). By hypothesis, C is a maximal commutative separable subalgebra of $V_B(B^H) \# H$, so $C \otimes_{C^H} (V_B(B^H) \# H) \cong \text{Hom}_C(V_B(B^H) \# H, V_B(B^H) \# H)$ ([4], Theorem 5.5, page 64). On the other hand, $V_B(B^H) \# H \cong \text{Hom}_{C^H}(V_B(B^H), V_B(B^H))$ ([7], theorem 3.4). Thus $C \otimes_{C^H} (V_B(B^H) \# H) \cong C \otimes_{C^H} \text{Hom}_{C^H}(V_B(B^H), V_B(B^H)) \cong \text{Hom}_C(C \otimes_{C^H} V_B(B^H), C \otimes_{C^H} V_B(B^H))$; and so $\text{Hom}_C(V_B(B^H) \# H, V_B(B^H) \# H) \cong \text{Hom}_C(C \otimes_{C^H} V_B(B^H), C \otimes_{C^H} V_B(B^H))$. This implies that $V_B(B^H) \# H \cong P \otimes_C (C \otimes_{C^H} V_B(B^H))$ for some finitely generated projective C -module P of rank 1, that is, $V_B(B^H) \# H \cong P \otimes_{C^H} V_B(B^H)$. Taking $\text{rank}_{C^H}(\)$ both sides, we have that $n \cdot \text{rank}_{C^H}(V_B(B^H)) = (\text{rank}_{C^H}(P)) \cdot (\text{rank}_{C^H}(V_B(B^H)))$ where $n = \dim_k(H)$. But $\text{rank}_{C^H}(V_B(B^H))$ is also n , so $\text{rank}_{C^H}(C) = \text{rank}_{C^H}(P) = n = \text{rank}_{C^H}(V_B(B^H))$.

Theorem 4.1 implies that the Azumaya C^H -algebras $V_B(B^H) \# H$ and $B \# H$ have a nice splitting ring C which is an H^* -Galois algebra over C^H and separable over C^H such that $C \otimes_{C^H} (V_B(B^H) \# H)$ and $C \otimes_{C^H} (B \# H)$ are matrix algebras.

COROLLARY 4.2. *If B is an H^* -DeMeyer-Kanzaki Galois extension of B^H , then $C \otimes_{C^H} (V_B(B^H) \# H) \cong M_n(C)$, the matrix algebra over C of order n where $n = \dim_k(H)$.*

PROOF. By hypothesis, B is an H^* -DeMeyer-Kanzaki Galois extension of B^H , so $C (= V_B(B^H))$ is an H^* -Galois algebra over C^H by Theorem 3.2. Hence C is a separable C^H -algebra and $C\#H$ is an Azumaya C^H -algebra ([7], Theorem 3.4). Thus C^H is a direct summand of C as a C^H -module. Therefore $C \otimes_{C^H} (C\#H) \cong M_n(C)$ by Theorem 3.4.

COROLLARY 4.3. *If B is an H^* -DeMeyer-Kanzaki Galois extension of B^H , then $C \otimes_{C^H} (B\#H) \cong M_n(B)$, the matrix algebra over B of order n where $n = \dim_k(H)$.*

PROOF. By Corollary 4.2, $C \otimes_{C^H} (C\#H) \cong M_n(C)$, so

$$B^H \otimes_{C^H} C \otimes_{C^H} (C\#H) \cong B^H \otimes_{C^H} M_n(C).$$

Since $B = B^H \cdot V_B(B^H) \cong B^H \otimes_{C^H} V_B(B^H) = B^H \otimes_{C^H} C$, we have that

$$\begin{aligned} C \otimes_{C^H} (B\#H) &\cong C \otimes_{C^H} ((B^H \otimes_{C^H} C)\#H) \\ &\cong C \otimes_{C^H} B^H \otimes_{C^H} (C\#H) \\ &\cong B^H \otimes_{C^H} C \otimes_{C^H} (C\#H) \\ &\cong B^H \otimes_{C^H} M_n(C) \cong M_n(B^H \otimes_{C^H} C) \\ &\cong M_n(B). \end{aligned}$$

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