On Weak Center Galois Extensions of Rings

by

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Abstract. Let B be a ring with 1, C the center of B, G a finite automorphism group of B, and B^G the set of elements in B fixed under each element in G. Then, the notion of a center Galois extension of B^G with Galois group G (i.e., C is a Galois algebra over C^G with Galois group $G|_C \cong G$) is generalized to a weak center Galois extension with group G, where B is called a weak center Galois extension with group G if $BI_i = Be_i$ for some idempotent in C and $I_i = \{c - g_i(c) \mid c \in C\}$ for each $g_i \neq 1$ in G. It is shown that B is a weak center Galois extension with group G if and only if for each $g_i \neq 1$ in G there exists an idempotent e_i in C and $\{b_k e_i \in Be_i; c_k e_i \in Ce_i,\ k = 1, 2, \cdots, m\}$ such that $\sum_{k=1}^{m} b_k e_i g_i(c_k e_i) = \delta_{1,g_i} e_i$ and g_i restricted to $C(1 - e_i)$ is an identity, and a structure of a weak center Galois extension with group G is also given.

Key Words and Phrases. Galois extensions, Center Galois extensions, Weak Center Galois Extensions, and Boolean algebras.

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1. Introduction. Galois theory for fields was generalized for rings in the sixties and seventies ([3], [4], [7], and [8]). Let B be a ring with 1, $G = \{g_1 = 1, g_2, \cdots, g_n\}$ an automorphism group of B of order n for some integer n, C the center of B, and B^G the set of elements in B fixed under each element in G. There are several well known classes of noncommutative Galois extensions: (1) the DeMeyer-Kanzaki Galois extension B (that is, B is an Azumaya C-algebra which is a Galois algebra with Galois group $G|_C \cong G$) ([3], [7]), (2) the H-separable extension B (that is, B is a Galois and a H-separable extension of B^G) ([8]), (3) the Azumaya Galois extension B (that is, B is a Galois extension of B^G which is an Azumaya C^G-algebra) ([1]), (4) the central Galois algebra ([3], [4], [7]), and (5) the center Galois extension B (that is, C is a Galois algebra over C^G with Galois group $G|_C \cong G$) ([11]). We note that a commutative Galois extension is a DeMeyer-Kanzaki Galois extension which is a center Galois extension. It is well know that C is a Galois extension of C^G if and only if the ideals generated by $\{c - g(c) \mid c \in C\}$ is C for each $g \neq 1$ in G ([2], Proposition 1.2, p.80). This fact was generalized in [11] to a center Galois extension; that is, B is a center Galois extension of B^G if and only if the ideals of B generated by $\{c - g(c) \mid c \in C\}$ is B, that is, $BI_i = B$ where $I_i = \{c - g_i(c) \mid c \in C\}$ for each $g_i \neq 1$ in G (for more about center Galois extensions, see [5], [6], [9], [10], and [11]). Generalizing the condition that $BI_i = B = B1$ to that $BI_i = Be_i$ for some idempotent e_i in C for each $g_i \neq 1$ in G, we obtain a broader class of rings B than the class of center Galois extensions. This class of rings are called weak center Galois extensions. The purpose of the present paper is to give a characterization and a structure of a weak center Galois extension B with group G. We shall show that B is a weak center Galois extension with group G if and only if for each $g_i \neq 1$ in G there exists an idempotent e_i in C and $\{b_k e_i \in Be_i; c_k e_i \in Ce_i,\, k = 1, 2, \cdots, m\}$ such that $\sum_{k=1}^{m} b_k e_i g_i(c_k e_i) = \delta_{1,g_i} e_i$ and g_i restricted to $C(1 - e_i)$ is an identity. Next, we call B a T-Galois extension of B^T if there exist elements $\{a_i, b_i$ in B, $i = 1, 2, ..., m\}$ for some integer m such that $\sum_{i=1}^{m} a_i g(b_i) = \delta_{1,g}$ for $g \in T \cup \{1\}$. We note that T is not necessarily a subgroup of G. Let B be a weak center Galois extension with group G. Then, we shall show that there exists a partition of $G - \{1\}$, $\{T_j \subset G,\, j = 1, 2, \cdots, h$ for some integer $h\}$ and some idempotents $e_j \in C$, $j = 1, 2, \cdots, h$ such that Be_j is a T_j-Galois extension of $(Be_j)^{T_j}$. So $B = \sum_{j=1}^{h} Be_j \oplus B(1 - \vee_{j=1}^{h} e_j)$ such that Be_j is a T_j-Galois extension of $(Be_j)^{T_j}$ for $j = 1, 2, \cdots, h$, where $\vee$ is the sum of the Boolean algebra of the idempotents in C. Moreover, when G is abelian, e_j can be taken as orthogonal idempotents in C so that $\sum_{j=1}^{h} Be_j$ is a direct sum. Furthermore, a sufficient condition is given for the existence of a subgroup $H_j \subset T_j \cup \{1\}$ for $j = 1, 2, \cdots, h$. In this case, Be_j is a H_j-Galois extension of $(Be_j)^{H_j}$ with Galois group H_j.

2. Definitions and Notations.

Throughout this paper, B will represent a ring with 1, $G = \{g_1 = 1, g_2, \cdots, g_n\}$ an automorphism group of B of order n for some integer n, C the center of B, and B^G the set of elements in B fixed under each element in G. We denote $I_i = \{c - g_i(c) \mid c \in C\}$ and BI_i the ideal of B generated by I_i for $g_i \in G$.

B is called a G-Galois extension of B^G if there exist elements $\{a_i, b_i$ in B, $i = 1, 2, ..., m\}$ for some integer m such that $\sum_{i=1}^{m} a_i g(b_i) = \delta_{1,g}$. Such a set $\{a_i, b_i\}$ is called a G-Galois system for B. B is called a weak center Galois extension of B^G with group G if

$BI_i = Be_i$ for some idempotent in C for each $g_i \neq 1$ in G. For a subset T (not necessary a subgroup) of G, B is called a T-Galois extension of B^T if there exist elements $\{a_i, b_i$ in B, $i = 1, 2, ..., m\}$ for some integer m such that $\sum_{i=1}^m a_i g(b_i) = \delta_{1,g}$ for $g \in T \cup \{1\}$. Such a set $\{a_i, b_i\}$ is called a T-Galois system for B. For a B-module M, we denote $\mathrm{Ann}_B(M) = \{b \in B \mid bm = 0 \text{ for all } m \in M\}$.

3. Weak Center Galois Extensions.

In [11], the present authors showed that a center Galois extension B is equivalent to each of the following statements: (i) $BI_i = B$ for each $g_i \neq 1$ in G and (ii) B is a Galois extension of B^G with a Galois system $\{b_i \in B,\ c_i \in C,\ i = 1, 2, ..., m\}$ for some integer m. In this section, we shall generalize this characterization to a weak center Galois extension B with group G. We begin with a Lemma.

Lemma 3.1. *If B is a weak center Galois extension with group G, then*
(1) g_i restricted to Be_i is an automorphism of Be_i.
(2) Be_i is a $\{g_i\}$-Galois extension of $(Be_i)^{\{g_i\}}$.

Proof. (1) For any $b = \sum_{k=1}^m b_k(c_k - g_i(c_k)) \in BI_i = Be_i$, where $b_k \in B$ and $c_k \in C$, $k = 1, 2, ..., m$ for some integer m, we have $g_i(b) = g_i(\sum_{k=1}^m b_k(c_k - g_i(c_k))) = \sum_{k=1}^m g_i(b_k)(g_i(c_k) - g_i(g_i(c_k))) \in BI_i = Be_i$. Hence $g_i(Be_i) \subset Be_i$. Thus, g_i restricted to Be_i is an automorphism of Be_i since g_i is an automorphism of B.

(2) Since $BI_i = Be_i$, there exist $\{b_k \in B,\ c_k \in C,\ k = 1, 2, ..., m\}$ for some integer m such that $\sum_{k=1}^m b_k(c_k - g_i(c_k)) = e_i$. Therefore, $\sum_{k=1}^m b_k c_k = e_i + \sum_{k=1}^m b_k g_i(c_k)$. Let $b_{m+1} = -\sum_{k=1}^m b_k g_i(c_k)$ and $c_{m+1} = 1$. Then $\sum_{k=1}^{m+1} b_k c_k = e_i$ and $\sum_{k=1}^{m+1} b_k g_i(c_k) = 0$. Noting that e_i is the identity of Be_i and g_i restricted to Be_i is an automorphism of Be_i, we have $g_i(e_i) = e_i$. Hence $\sum_{k=1}^{m+1} b_k e_i g_i(c_k e_i) = \delta_{1,g_i} e_i$, that is, $\{b_k e_i; c_k e_i,\ k = 1, 2, \cdots, m+1\}$ is a $\{g_i\}$-Galois system for Be_i.

Next is an equivalent condition for a weak center Galois extension with group G.

Theorem 3.2. *B is a weak center Galois extension with group G (that is, $BI_i = Be_i$ for some idempotent e_i in C for each $g_i \neq 1$ in G) if and only if for each $g_i \neq 1$ in G there exists an idempotent e_i in C and $\{b_k e_i \in Be_i; c_k e_i \in Ce_i,\ k = 1, 2, \cdots, m\}$ such that $\sum_{k=1}^{m} b_k e_i g_i(c_k e_i) = \delta_{1,g_i} e_i$ and g_i restricted to $C(1 - e_i)$ is an identity.*

Proof. $(\Longrightarrow)$: By Lemma 3.1-(2), BI_i $(= Be_i)$ contains a $\{g_i\}$-Galois system $\{b_k e_i \in Be_i; c_k e_i \in Ce_i,\ k = 1, 2, \cdots, m\}$ such that $\sum_{k=1}^{m} b_k e_i g_i(c_k e_i) = \delta_{1,g_i} e_i$. Next we show that g_i restricted to $C(1 - e_i)$ is an identity. In fact, by Lemma 3.1-(1), $g_i(e_i) = e_i$. Hence for any $c \in C$, $c(1 - e_i) - g_i(c(1 - e_i)) = (c - g_i(c))(1 - e_i) \in Ce_i \cap C(1 - e_i) = \{0\}$. Thus, $g_i(c(1 - e_i)) = c(1 - e_i)$ for all $c \in C$. This proves that g_i restricted to $C(1 - e_i)$ is an identity.

$(\Longleftarrow)$: By hypothesis, for each $g_i \neq 1$ in G there exists an idempotent e_i in C and $\{b_k e_i \in Be_i; c_k e_i \in Ce_i,\ k = 1, 2, \cdots, m\}$ such that $\sum_{k=1}^{m} b_k e_i g_i(c_k e_i) = \delta_{1,g_i} e_i$. Hence $e_i = \sum_{k=1}^{m} b_k e_i (c_k e_i - g_i(c_k e_i)) \in BI_i$. Hence $Be_i \subset BI_i$. But e_i is an idempotent, so $Be_i = Be_i e_i \subset BI_i e_i \subset Be_i$. Thus, $Be_i = BI_i e_i$. Since g_i restricted to $C(1 - e_i)$ is an identity, $g_i(c(1 - e_i)) = c(1 - e_i)$ for all $c \in C$ (in particular, $g_i(e_i) = e_i$). Hence $c - g_i(c) = ce_i - g_i(ce_i) = (c - g_i(c))e_i$ for all $c \in C$. This implies that $Be_i = BI_i e_i = BI_i$.

Recall that B is called a T-Galois extension of B^T for a subset T (not necessary a subgroup) of G if B contains a T-Galois system. Next we give a structure of a weak center Galois extension with group G.

Lemma 3.3. *Assume B is a weak center Galois extension with group G. Let $T_j = \{g_i \in G \,|\, BI_i = Be_j$, that is, $e_i = e_j\}$. Then Be_j is a T_j-Galois extension of $(Be_j)^{T_j}$ for each $j \neq 1$.*

Proof. By the proof of Lemma 3.1-(2), for each $g_i \in T_j$, there is a $\{g_i\}$-Galois system $\{b_k^{(i)} e_j; c_k^{(i)} e_j,\ k = 1, 2, \cdots, m_i\}$ for Be_j where $b_k^{(i)} \in B$ and $c_k^{(i)} \in C$, $k = 1, 2, \cdots, m_i$ for some integer m_i. Denote the elements in T_j by $\{g_{i_1}, g_{i_2}, \cdots, g_{i_t}\}$ for some integer t. Let $b_{k_1,k_2,\cdots,k_t} = b_{k_1}^{(i_1)} b_{k_2}^{(i_2)} \cdots b_{k_t}^{(i_t)} e_j$ and $c_{k_1,k_2,\cdots,k_t} = c_{k_1}^{(i_1)} c_{k_2}^{(i_2)} \cdots c_{k_t}^{(i_t)} e_j$ for $k_l = 1, 2, \cdots m_{i_l}$ and $l = 1, 2, \cdots t$. Noting that $c_{k_l}^{(i_l)} \in C$, $l = 1, 2, \cdots t$, we have

$$\sum_{k_1=1}^{m_{i_1}}\sum_{k_1=2}^{m_{i_2}}\cdots\sum_{k_t=1}^{m_{i_t}} b_{k_1,k_2,\cdots,k_t}c_{k_1,k_2,\cdots,k_t}$$
$$= \sum_{k_1=1}^{m_{i_1}}\sum_{k_1=2}^{m_{i_2}}\cdots\sum_{k_t=1}^{m_{i_t}} (b_{k_1}^{(i_1)}b_{k_2}^{(i_2)}\cdots b_{k_t}^{(i_t)}e_j)(c_{k_1}^{(i_1)}c_{k_2}^{(i_2)}\cdots c_{k_t}^{(i_t)}e_j)$$
$$= \sum_{k_1=1}^{m_{i_1}}(b_{k_1}^{(i_1)}e_j)(c_{k_1}^{(i_1)}e_j)\sum_{k_1=2}^{m_{i_2}}(b_{k_2}^{(i_2)}e_j)(c_{k_2}^{(i_2)}e_j)\cdots\sum_{k_t=1}^{m_{i_t}}(b_{k_t}^{(i_t)}e_j)(c_{k_t}^{(i_t)}e_j)$$
$$= e_j,$$

and for each $g_i \in T_j$

$$\sum_{k_1=1}^{m_{i_1}}\sum_{k_1=2}^{m_{i_2}}\cdots\sum_{k_t=1}^{m_{i_t}} b_{k_1,k_2,\cdots,k_t}g_i(c_{k_1,k_2,\cdots,k_t})$$
$$= \sum_{k_1=1}^{m_{i_1}}\sum_{k_1=2}^{m_{i_2}}\cdots\sum_{k_t=1}^{m_{i_t}} (b_{k_1}^{(i_1)}b_{k_2}^{(i_2)}\cdots b_{k_t}^{(i_t)}e_j)g_i(c_{k_1}^{(i_1)}c_{k_2}^{(i_2)}\cdots c_{k_t}^{(i_t)}e_j)$$
$$= \sum_{k_1=1}^{m_{i_1}}(b_{k_1}^{(i_1)}e_j)g_i(c_{k_1}^{(i_1)}e_j)\sum_{k_1=2}^{m_{i_2}}(b_{k_2}^{(i_2)}e_j)g_i(c_{k_2}^{(i_2)}e_j)\cdots\sum_{k_t=1}^{m_{i_t}}(b_{k_t}^{(i_t)}e_j)g_i(c_{k_t}^{(i_t)}e_j)$$
$$= 0.$$

Thus, $\{b_{k_1,k_2,\cdots,k_t};\ c_{k_1,k_2,\cdots,k_t},\ k_l = 1,2,\cdots m_{i_l}$ and $l = 1,2,\cdots t\}$ is a T_j-Galois system for Be_j. This completes the proof.

Theorem 3.4. *If B is a weak center Galois extension with group G, then there exists a partition $\{T_j \subset G,\ j = 1,2,\cdots,m\}$ of $G - \{1\}$ and a finite set of central idempotents $\{e'_i | i = 1,2,\cdots,m$ for some integer $m\}$ such that (1) Be'_j is a T_j-Galois extension of $(Be'_j)^{T_j}$, (2) $B = \sum_{j=1}^m Be'_j \oplus B(1 - \vee_{j=1}^m e'_j)$ where $\vee_{j=1}^m e'_j$ is the sum of $e'_1, e'_2, \cdots, e'_m$ in the Boolean algebra of all idempotents in C, and (3) $G|_{C(1-\vee_{j=1}^m e'_j)} = \{1\}$*

Proof. (1) Since $BI_i = Be_i$ for some idempotent e_i in C for each $g_i \neq 1$ in G, we have a set of central idempotents $\{e_i | g_i \neq 1 \text{ in } G\}$. Let $E = \{e'_j | j = 1,2,\cdots,m\}$ be the set of all distinct idempotents in $\{e_i | g_i \neq 1 \text{ in } G\}$ and let $T_j = \{g_i \in G \,|\, BI_i = Be'_j$, that is, $e_i = e'_j\}$. Then Be'_j is a T_j-Galois extension of $(Be'_j)^{T_j}$ for each $j = 1,2,\cdots,m$ by Lemma 3.2. Moreover, since $E = \{e'_j | j = 1,2,\cdots,m\}$ is the set of all distinct idempotents in $\{e_i | BI_i = Be_i \text{ for } g_i \neq 1 \text{ in } G\}$, it is easy to see that $T_i \cap T_j = \emptyset$, the empty set for $i \neq j$ and that $\cup_{j=1}^m T_j = G - \{1\}$, that is, $\{T_j \subset G,\ j = 1,2,\cdots,m\}$ is a partition of $G - \{1\}$.

Part (2) is an immediate consequence of Part (1), and Theorem 3.2 implies part (3).

We remark that the partition of $G-\{1\}$, $\{T_j \subset G,\ j=1,2,\cdots,m\}$ is determined by the set of all distinct idempotents in $\{e_i | BI_i = Be_i \text{ for } g_i \neq 1 \text{ in } G\}$.

When G is abelian, we shall obtain a stronger structure of a weak center Galois extension with group G.

Lemma 3.5. *Assume B is a weak center Galois extension with group G. If G is abelian, then $g_j(e_i)=e_i$ for all $i,j=2,3,\cdots,n$.*

Proof. For any $c-g_i(c)\in I_i$, $g_j(c-g_i(c)) = g_j(c)-g_i(g_j(c))\in I_i$. Hence $g_j(BI_i)\subset BI_i$. Thus, g_j restricted to BI_i $(=Be_i)$ is an automorphism of Be_i since g_j is an automorphism of B. Therefore, $g_j(e_i)=e_i$.

Theorem 3.6. *Assume B is a weak center Galois extension with group G. If G is abelian, then there exist orthogonal idempotents $\{f_i | i=1,2,\cdots,p \text{ for some integer } p\}$ and some subset $T^{(i)}$ of G, $i=1,2,\cdots,p$ such that $B=\oplus\sum_{i=1}^{p}Bf_i\oplus B(1-\vee_{i=1}^{p}f_i)$ where $\vee_{i=1}^{p}f_i$ is the sum of $f_1,f_2,\cdots,f_p$ in the Boolean algebra of all idempotents in C and Bf_i is a $T^{(i)}$-Galois extension of $(Bf_i)^{T^{(i)}}$ for $i=1,2,\cdots,p$.*

Proof. By Theorem 3.4, there exists a set of distinct idempotents $E=\{e'_j | j=1,2,\cdots,m\}$ in C and a partition $\{T_j | j=1,2,\cdots,m\}$ of $G-\{1\}$ such that Be'_j is a T_j-Galois extension of $(Be'_j)^{T_j}$ for $j=1,2,\cdots,m$. Now let S be the Boolean subalgebra generated by E with all non-zero minimal elements $f_1,f_2,\cdots,f_p$ in S. Then, it is easy to see that $f_if_j=0$ for $i\neq j$, and so $f_1,f_2,\cdots,f_p$ are orthogonal idempotents in C. For each f_i, $i=1,2,\cdots,p$, $f_i=e'_{j_1}e'_{j_2}\cdots e'_{j_{p_i}}$. By Theorem 3.4, Be'_{j_l} is a T_{j_l}-Galois extension of $(Be'_{j_l})^{T_{j_l}}$ for each $l=1,2,\cdots,p_i$ with a T_{j_l}-Galois system $\{b^{(l)}_{t_l}e'_{j_l}; c^{(l)}_{t_l}e'_{j_l} \mid b^{(l)}_{t_l}\in B, c^{(l)}_{t_l}\in C$, and $t_l=1,2,\cdots,m_l\}$. Hence, by using the same patching method as given in Lemma 3.2, $\{b_{t_1,t_2,\cdots,t_{p_i}} = b^{(1)}_{t_1}b^{(2)}_{t_2}\cdots b^{(p_i)}_{t_{p_i}}f_i;\ c_{t_1,t_2,\cdots,t_{p_i}} = c^{(1)}_{t_1}c^{(2)}_{t_2}\cdots c^{(p_i)}_{t_{p_i}}f_i \mid t_l=1,2,\cdots,m_l$ and $l=1,2,\cdots p_i\}$ is a $T^{(i)}$-Galois system for Bf_i where $T^{(i)}=\cup_{l=1}^{k_i}T_{j_l}$. Thus, $B=\oplus\sum_{i=1}^{p}Bf_i\oplus B(1-\vee_{i=1}^{p}f_i)$ such that Bf_i is a $T^{(i)}$-Galois extension of $(Bf_i)^{T^{(i)}}$ for $i=1,2,\cdots,p$ and $\{f_1,f_2,\cdots,f_p\}$ is a set of orthogonal idempotents in C.

4. Special Cases.

We note that the T_i's in Theorem 3.4 and $T^{(i)}$'s in Theorem 3.6 may not be subgroups of G. Next, we give a sufficient condition for each $T_i\cup\{1\}$ $(\subset G)$

containing a subgroup H_i so that Be_i is a H_i-Galois extension of $(Be_i)^{H_i}$ with Galois group H_i. Consequently, Be_i becomes a center Galois extension of $(Be_i)^{H_i}$ with Galois group H_i, and B is a center Galois extension of G with Galois group G if $e_i = 1$ for all $g_i \neq 1$. We first show a relation between $B(1-e_p)$, $B(1-e_q)$, and $B(1-e_t)$ where $g_p g_q = g_t \in G$.

Lemma 4.1. *Let $J_i = \{b \in B \,|\, bc = g_i(c)b$ for all $c \in C\}$ for each $g_i \in G$. Then, $J_p J_q \subset J_t$ if $g_p g_q = g_t \in G$.*

Proof. Let $a \in J_p$ and $b \in J_q$. Then, for any $c \in C$, $(ab)c = ag_q(c)b = g_p(g_q(c))ab = g_t(c)(ab)$ where $g_p g_q = g_t$. Hence $ab \in J_t$. Thus, $J_p J_q \subset J_t$

Corollary 4.2. *If B is a weak center Galois extension with group G, then $B(1-e_p)B(1-e_q) \subset B(1-e_t)$ where $g_p g_q = g_t \in G$.*

Proof. Since B is a weak center Galois extension with group G, $BI_i = Be_i$ for some idempotent e_i in C for each $g_i \neq 1$ in G. But $I_i = \{c - g_i(c) \mid c \in C\}$, so $J_i = \{b \in B \,|\, bc = g_i(c)b$ for all $c \in C\} = \{b \in B \,|\, b(c - g_i(c)) = 0$ for all $c \in C\}$. Hence $J_i = Ann_B(I_i) = Ann_B(BI_i) = Ann_B(Be_i) = B(1-e_i)$. Thus, by Lemma 4.1, we have $B(1-e_p)B(1-e_q) \subset B(1-e_t)$ where $g_p g_q = g_t \in G$.

Theorem 4.3. *Assume B is a weak center Galois extension with group G. Let T_i for each $i = 2, 3, \cdots n$, be the subset of G as given in Theorem 3.4 such that Be_i is a T_i-Galois extension of $(Be_i)^{T_i}$, S the Boolean subalgebra generated by $\{e_i | g_i \neq 1$ in $G\}$ with all nonzero minimal elements $\{f_1, f_2, \cdots, f_k\}$ in S, and $H_j = \{1\} \cup \{g_i \in G \,|\, e_i f_j = f_j$ and $e_i f_l = 0$ for all $l \neq j\}$. Then, H_j is a subgroup of G for each of $j = 1, 2, \cdots, k$ such that $g_i(f_j) = f_j$ for each $g_i \in H_j$.*

Proof. (1) For any g_p and g_q in H_j, let $g_p g_q = g_t$ for some $g_t \in G$. We claim that $g_t \in H_j$ if $g_t \neq 1$. Since $g_t \neq 1$, $BI_t = Be_t$ for some idempotent $e_t \neq 0$ in C. By Corollary 4.2, $B(1-e_p)B(1-e_q) \subset B(1-e_t)$. Therefore, in the Boolean algebra of all idempotents in C with operations $\wedge$, $\vee$, complement, and the relation $<$, $(1-e_p)(1-e_q) < (1-e_t)$. So $e_t < e_p \vee e_q = e_p + e_q - e_p e_q$. Thus, $e_t = e_t(e_p + e_q - e_p e_q)$. Since $g_p, g_q \in H_j$, $e_p f_l = 0$ and $e_q f_l = 0$ for all $l \neq j$. Hence $e_t f_l = e_t(e_p + e_q - e_p e_q) f_l = 0$ for all $l \neq j$. Moreover, since S is the Boolean subalgebra generated by $\{e_i | g_i \neq 1$ in $G\}$, there is at least one nonzero minimal element in S less than e_t. But $e_t f_l = 0$ for all $l \neq j$, so f_j must be less than e_t. Hence $e_t f_j = f_j$. Thus, $g_t (= g_p g_q) \in H_j$, and so H_j is a subgroup of G. Moreover, suppose

$g_i \in H_j$. Then $e_i f_j = f_j$ and $e_i f_l = 0$ for all $l \neq j$. Hence, e_i is greater than f_j, but not greater than f_l for all $l \neq j$. Since $g_i(e_i) = e_i$, $g_i(f_j)$ is a nonzero minimal element in S less than e_i. Thus, $g_i(f_j) = f_j$.

Corollary 4.4. *Keeping the notation in Theorem 4.3, if* $H_j \neq \{1\}$ *for* $j = 1, 2, \cdots, p$, *then* $B = \oplus \sum_{j=1}^{p} Bf_j \oplus B(1 - \vee_{j=1}^{p} f_j)$, *where* $\vee_{j=1}^{p} f_j$ *is the sum of* $f_1, f_2, \cdots, f_p$ *in the Boolean algebra of all idempotents in* C, *such that* Bf_j *is a* H_j*-Galois extension of* $(Bf_j)^{H_j}$ *with Galois group* H_j *for* $j = 1, 2, \cdots, p$.

Corollary 4.5. *If* $BI_j = B$ *for each* $g_j \neq 1$ *in* G, *then* B *is a center Galois extension of* B^G *with Galois group* G.

Proof. Since $e_2 = e_3 = \cdots = e_n$, $T_2 = T_3 = \cdots = T_n = G - \{1\}$, so $T_j \cup \{1\} = G$. Thus, B is a Galois extension of B^G with a Galois system $\{b_i \in B; c_i \in C, i = 1, 2, \cdots, m\}$ for some integer m, that is, B is a center Galois extension of B^G with Galois group G.

If the order of each non-identity element in G has order 2 (hence G is abelian), the following theorem shows that $T_i \cup \{1\}$ contains a subgroup of G for each $g_j \neq 1$ in T_i.

Theorem 4.6. *Assume* B *is a weak center Galois extension with group* G. *If each non-identity element* g_i *in* G *has order 2, then* T_i *contains a subgroup of* H_i *of order 2 for each* $g_j \neq 1$ *in* G *such that* Be_i *is a* H_i*-Galois extension of* $(Be_i)^{H_i}$ *with Galois group* H_i.

Proof. Let $BI_i = Be_i$ for $g_i \neq 1$ in G. Then $H_i = \{1, g_i\}$ is a subgroup contained in $T_i \cup \{1\}$ where $T_i = \{g_k \in G \,|\, BI_k = Be_i\}$ as defined in Theorem 3.4. Since Be_i is a T_i-Galois extension of $(Be_i)^{T_i}$, Be_i is a H_i-Galois extension of $(Be_i)^{H_i}$ with Galois group H_i.

Theorem 3.4 shows that a weak center Galois extension is a sum of T_i-Galois extensions for some $T_i \subset G$ and Theorem 4.6 states a weak center Galois extension as a direct sum of center Galois extensions. The following is an example of a weak center Galois extension with group G as given in Theorem 4.6, but not a Galois extension.

Example. Let Q be the rational field, $B = Q \oplus Q \oplus Q \oplus Q \oplus Q$, and $G =$

$\{g_1 = 1, g_2, g_3, g_4 = g_2g_3\}$ such that $g_2(a_1, a_2, a_3, a_4, a_5) = (a_2, a_1, a_3, a_4, a_5)$ and $g_3(a_1, a_2, a_3, a_4, a_5) = (a_1, a_2, a_4, a_3, a_5)$ for all $(a_1, a_2, a_3, a_4, a_5) \in B$. Then,

(1) $BI_i = Be_i$ for each $g_i \neq 1$ in G, where $e_2 = (1, 1, 0, 0, 0)$, $e_3 = (0, 0, 1, 1, 0)$, and $e_4 = (1, 1, 1, 1, 0)$. Hence, B is a weak center Galois extension with group G.

(2) B is not a Galois extension since G restricted to $\{(0, 0, 0, 0, a) \,|\, a \in Q\}$ is identity.

(3) Let $H_i = \{1, g_i\}$, $i = 2, 3, 4$. Then H_i is a subgroup of G of order 2. Moreover, $BI_2 = Be_2$ is a center H_2-Galois extension of $(Be_2)^{H_2}$ with Galois system $\{b_1 = (1, 0, 0, 0, 0), b_2 = (0, 1, 0, 0, 0); c_1 = (1, 0, 0, 0, 0), c_2 = (0, 1, 0, 0, 0)\}$, $BI_3 = Be_3$ is a center H_3-Galois extension of $(Be_3)^{H_3}$ with Galois system $\{b_1 = (0, 0, 1, 0, 0), b_2 = (0, 0, 0, 1, 0); c_1 = (0, 0, 1, 0, 0), c_2 = (0, 0, 0, 1, 0)\}$, and $BI_4 = Be_4$ is a center H_4-Galois extension of $(Be_4)^{H_4}$ with Galois system $\{b_1 = (1, 0, 0, 0, 0), b_2 = (0, 1, 0, 0, 0), b_3 = (0, 0, 1, 0, 0), b_4 = (0, 0, 0, 1, 0); c_1 = (1, 0, 0, 0, 0), c_2 = (0, 1, 0, 0, 0), c_3 = (0, 0, 1, 0, 0), c_4 = (0, 0, 0, 1, 0)\}$.

(4) $S = \{0 = (0, 0, 0, 0, 0), e_2, e_3, e_4, 1 = (1, 1, 1, 1, 1)\}$ is the Boolean subalgebra generated by $E = \{e_2, e_3, e_4\}$ in the Boolean algebra of all idempotents in the center of B. The minimal elements in S are $f_1 = e_2$ and $f_2 = e_3$, and $f_1 \vee f_2 = e_4$. We have that $Bf_1 = \{(a_1, a_2, 0, 0, 0) \,|\, a_1, a_2 \in Q\}$, $Bf_2 = \{(0, 0, a_3, a_4, 0) \,|\, a_3, a_4 \in Q\}$, and $B(1 - f_1 \vee f_2) = \{(0, 0, 0, 0, a_5) \,|\, a_5 \in Q\}$. So $B = Bf_1 \oplus Bf_2 \oplus B(1 - f_1 \vee f_2)$ such that and Bf_j is a H_j-Galois extension of $(Bf_j)^{H_j}$ for $j = 1, 2$.

(5) Since $e_2 = (1, 1, 0, 0, 0)$, $e_3 = (0, 0, 1, 1, 0)$, and $e_4 = (1, 1, 1, 1, 0)$, we have $C(1 - e_2) = \{(0, 0, a_3, a_4, a_5) \,|\, a_3, a_4, a_5 \in Q\}$, $C(1 - e_3) = \{(a_1, a_2, 0, 0, a_5) \,|\, a_1, a_2, a_5 \in Q\}$, and $C(1 - e_4) = \{(0, 0, 0, 0, a_5) \,|\, a_5 \in Q\}$. So g_i restricted to $C(1 - e_i)$ is an identity for each $g_i \neq 1$ in G.

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