

# On Characterizations of a Center Galois Extension

by

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**Abstract.** Let  $B$  be a ring with 1,  $C$  the center of  $B$ ,  $G$  a finite automorphism group of  $B$ , and  $B^G$  the set of elements in  $B$  fixed under each element in  $G$ . Then, it is shown that  $B$  is a center Galois extension of  $B^G$  (that is,  $C$  is a Galois algebra over  $C^G$  with Galois group  $G|_C \cong G$ ) if and only if the ideal of  $B$  generated by  $\{c - g(c) \mid c \in C\}$  is  $B$  for each  $g \neq 1$  in  $G$ . This generalizes the well known characterization of a commutative Galois extension  $C$  that  $C$  is a Galois extension of  $C^G$  with Galois group  $G$  if and only if the ideal generated by  $\{c - g(c) \mid c \in C\}$  is  $C$  for each  $g \neq 1$  in  $G$ . Some more characterizations of a center Galois extension  $B$  are also given.

**Key Words and Phrases.** Galois extensions, Center Galois extensions, Central extensions, Galois central extensions, Azumaya algebras, Separable extensions, and  $H$ -separable extensions.

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**1. Introduction.** Let  $C$  be a commutative ring with 1,  $G$  a finite automorphism group of  $C$ , and  $C^G$  the set of elements in  $C$  fixed under each element in  $G$ . It is well known that a commutative Galois extension  $C$  is characterized in terms of the ideals generated by  $\{c - g(c) \mid c \in C\}$  for  $g \neq 1$  in  $G$ , that is,  $C$  is a Galois extension with Galois group  $G$  if and only if the ideal generated by  $\{c - g(c) \mid c \in C\}$  is  $C$  for each  $g \neq 1$  in  $G$  ([3], Proposition 1.2, p.80). A natural generalization of a commutative Galois extension is the notion of a center Galois extension, that is, a noncommutative ring  $B$  with a finite automorphism group  $G$  and center  $C$  is called a center Galois extension of  $B^G$  with Galois group  $G$  if  $C$  is a Galois extension of  $C^G$  with Galois group  $G|_C \cong G$ . S. Ikehata ([4],[5]) characterized a center Galois extension with a cyclic Galois group  $G$  of prime order in terms of a skew

polynomial ring. Then, the present authors generalized the Ikehata characterization to center Galois extensions with Galois group  $G$  of any cyclic order ([7]) and to center Galois extensions with any finite Galois group  $G$  ([8]). The purpose of the present paper is to generalize the above characterization of a commutative Galois extension to a center Galois extension. We shall show that  $B$  is a center Galois extension of  $B^G$  if and only if the ideal of  $B$  generated by  $\{c - g(c) \mid c \in C\}$  is  $B$  for each  $g \neq 1$  in  $G$ . A center Galois extension  $B$  is also equivalent to each of the following statements: (i)  $B$  is a Galois central extension of  $B^G$ , that is,  $B = B^G C$  which is a  $G$ -Galois extension of  $B^G$ . (ii)  $B$  is a Galois extension of  $B^G$  with a Galois system  $\{b_i \in B, c_i \in C, i = 1, 2, \dots, m\}$  for some integer  $m$ , and (iii) the ideal of the subring  $B^G C$  generated by  $\{c - g(c) \mid c \in C\}$  is  $B^G C$  for each  $g \neq 1$  in  $G$ .

**2. Definitions and Notations.** Throughout this paper,  $B$  will represent a ring with 1,  $G = \{g_1 = 1, g_2, \dots, g_n\}$  an automorphism group of  $B$  of order  $n$  for some integer  $n$ ,  $C$  the center of  $B$ ,  $B^G$  the set of elements in  $B$  fixed under each element in  $G$ , and  $B * G$  a skew group ring in which the multiplication is given by  $gb = g(b)g$  for  $b \in B$  and  $g \in G$ .

$B$  is called a  $G$ -Galois extension of  $B^G$  if there exist elements  $\{a_i, b_i$  in  $B, i = 1, 2, \dots, m\}$  for some integer  $m$  such that  $\sum_{i=1}^m a_i g(b_i) = \delta_{1,g}$ . Such a set  $\{a_i, b_i\}$  is called a  $G$ -Galois system for  $B$ .  $B$  is called a center Galois extension of  $B^G$  if  $C$  is a Galois algebra over  $B^G$  with Galois group  $G|_C \cong G$ .  $B$  is called a central extension of  $B^G$  if  $B = B^G C$ , and  $B$  is called a Galois central extension of  $B^G$  if  $B = B^G C$  is a Galois extension of  $B^G$  with Galois group  $G$ .

Let  $A$  be a subring of a ring  $B$  with the same identity 1. We denote  $V_B(A)$  the commutator subring of  $A$  in  $B$ . We call  $B$  a separable extension of  $A$  if there exist  $\{a_i, b_i$  in  $B, i = 1, 2, \dots, m$  for some integer  $m\}$  such that  $\sum a_i b_i = 1$ , and  $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$  for all  $b$  in  $B$  where  $\otimes$  is over  $A$ .  $B$  is called a  $H$ -separable extension of  $A$  if  $B \otimes_A B$  is isomorphic to a direct summand of a finite direct sum of  $B$  as a  $B$ -bimodule.  $B$  is called centrally projective over  $A$  if  $B$  is a direct summand of a finite direct sum of  $A$  as a  $A$ -bimodule.

**3. The Characterizations.** In this section, we denote  $J_j^{(C)} = \{c - g_j(c) \mid c \in C\}$ . We shall show that  $B$  is a center Galois extension of  $B^G$  if and only if  $B = B J_j^{(C)}$ ,

the ideal of  $B$  generated by  $J_j^{(C)}$ , for each  $g_j \neq 1$  in  $G$ . Some more characterizations of a center Galois extension  $B$  are also given. We begin with a lemma.

**LEMMA 3.1.** *If  $B = BJ_j^{(C)}$  for each  $g_j \neq 1$  in  $G$  (that is,  $j \neq 1$ ), then*

(1)  *$B$  is a Galois extension of  $B^G$  with Galois group  $G$  and a Galois system  $\{b_i \in B; c_i \in C, i = 1, 2, \dots, m\}$  for some integer  $m$ .*

(2)  *$B$  is centrally projective over  $B^G$ .*

(3)  *$B * G$  is  $H$ -separable over  $B$ .*

(4)  *$V_{B * G}(B) = C$ .*

PROOF. (1) Since  $B = BJ_j^{(C)}$  for each  $j \neq 1$  in  $G$ , there exist  $\{b_i^{(j)} \in B, c_i^{(j)} \in C, i = 1, 2, \dots, m_j\}$  for some integer  $m_j, j = 2, 3, \dots, n$  such that  $\sum_{i=1}^{m_j} b_i^{(j)}(c_i^{(j)} - g_j(c_i^{(j)})) = 1$ . Therefore,  $\sum_{i=1}^{m_j} b_i^{(j)} c_i^{(j)} = 1 + \sum_{i=1}^{m_j} b_i^{(j)} g_j(c_i^{(j)})$ . Let  $b_{m_j+1}^{(j)} = -\sum_{i=1}^{m_j} b_i^{(j)} g_j(c_i^{(j)})$  and  $c_{m_j+1}^{(j)} = 1$ . Then  $\sum_{i=1}^{m_j+1} b_i^{(j)} c_i^{(j)} = 1$  and  $\sum_{i=1}^{m_j+1} b_i^{(j)} g_j(c_i^{(j)}) = 0$ . Let  $b_{i_2, i_3, \dots, i_n} = b_{i_2}^{(2)} b_{i_3}^{(3)} \cdots b_{i_n}^{(n)}$  and  $c_{i_2, i_3, \dots, i_n} = c_{i_2}^{(2)} c_{i_3}^{(3)} \cdots c_{i_n}^{(n)}$  for  $i_j = 1, 2, \dots, m_j + 1$  and  $j = 2, 3, \dots, n$ . Then

$$\begin{aligned} & \sum_{i_2=1}^{m_2+1} \sum_{i_3=1}^{m_3+1} \cdots \sum_{i_n=1}^{m_n+1} b_{i_2, i_3, \dots, i_n} c_{i_2, i_3, \dots, i_n} \\ &= \sum_{i_2=1}^{m_2+1} \sum_{i_3=1}^{m_3+1} \cdots \sum_{i_n=1}^{m_n+1} b_{i_2}^{(2)} b_{i_3}^{(3)} \cdots b_{i_n}^{(n)} c_{i_2}^{(2)} c_{i_3}^{(3)} \cdots c_{i_n}^{(n)} \\ &= \sum_{i_2=1}^{m_2+1} \sum_{i_3=1}^{m_3+1} \cdots \sum_{i_n=1}^{m_n+1} b_{i_2}^{(2)} c_{i_2}^{(2)} b_{i_3}^{(3)} c_{i_3}^{(3)} \cdots b_{i_n}^{(n)} c_{i_n}^{(n)} \\ &= \sum_{i_2=1}^{m_2+1} b_{i_2}^{(2)} c_{i_2}^{(2)} \sum_{i_3=1}^{m_3+1} b_{i_3}^{(3)} c_{i_3}^{(3)} \cdots \sum_{i_n=1}^{m_n+1} b_{i_n}^{(n)} c_{i_n}^{(n)} \\ &= 1 \end{aligned}$$

and for each  $j \neq 1$

$$\begin{aligned} & \sum_{i_2=1}^{m_2+1} \sum_{i_3=1}^{m_3+1} \cdots \sum_{i_n=1}^{m_n+1} b_{i_2, i_3, \dots, i_n} g_j(c_{i_2, i_3, \dots, i_n}) \\ &= \sum_{i_2=1}^{m_2+1} \sum_{i_3=1}^{m_3+1} \cdots \sum_{i_n=1}^{m_n+1} b_{i_2}^{(2)} b_{i_3}^{(3)} \cdots b_{i_n}^{(n)} g_j(c_{i_2}^{(2)} c_{i_3}^{(3)} \cdots c_{i_n}^{(n)}) \\ &= \sum_{i_2=1}^{m_2+1} \sum_{i_3=1}^{m_3+1} \cdots \sum_{i_n=1}^{m_n+1} b_{i_2}^{(2)} b_{i_3}^{(3)} \cdots b_{i_n}^{(n)} g_j(c_{i_2}^{(2)}) g_j(c_{i_3}^{(3)}) \cdots g_j(c_{i_n}^{(n)}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i_2=1}^{m_2+1} \sum_{i_3=1}^{m_3+1} \cdots \sum_{i_n=1}^{m_n+1} b_{i_2}^{(2)} g_j(c_{i_2}^{(2)}) b_{i_3}^{(3)} g_j(c_{i_3}^{(3)}) \cdots b_{i_n}^{(n)} g_j(c_{i_n}^{(n)}) \\
&= \sum_{i_2=1}^{m_2+1} b_{i_2}^{(2)} g_j(c_{i_2}^{(2)}) \sum_{i_3=1}^{m_3+1} b_{i_3}^{(3)} g_j(c_{i_3}^{(3)}) \cdots \sum_{i_n=1}^{m_n+1} b_{i_n}^{(n)} g_j(c_{i_n}^{(n)}) \\
&= 0
\end{aligned}$$

Thus,  $\{b_{i_2, i_3, \dots, i_n} \in B; c_{i_2, i_3, \dots, i_n} \in C, i_j = 1, 2, \dots, m_j + 1 \text{ and } j = 2, 3, \dots, n\}$  is a Galois system for  $B$ . This completes the proof of (1).

(2) By (1),  $B$  is a Galois extension of  $B^G$  with a Galois system  $\{b_i \in B, c_i \in C, i = 1, 2, \dots, m\}$  for some integer  $m$ . Let  $f_i : B \rightarrow B^G$  given by  $f_i(b) = \sum_{j=1}^n g_j(c_i b)$  for all  $b \in B, i = 1, 2, \dots, m$ . Then, it is easy to check that  $f_i$  is a homomorphism as  $B^G$ -bimodule and  $b = \sum_{i=1}^m b_i c_i b = \sum_{j=1}^n \sum_{i=1}^m b_i g_j(c_i) g_j(b) = \sum_{i=1}^m b_i \sum_{j=1}^n g_j(c_i b) = \sum_{i=1}^m b_i f_i(b)$  for all  $b \in B$ . Hence  $\{b_i; f_i, i = 1, 2, \dots, m\}$  is a dual bases for  $B$  as  $B^G$ -bimodule, and so  $B$  is finitely generated and projective as  $B^G$ -bimodule. Therefore,  $B$  is a direct summand of a finite direct sum of  $B^G$  as a  $B^G$ -bimodule. Thus  $B$  is centrally projective over  $B^G$ .

(3) By (1),  $B$  is a Galois extension of  $B^G$  with Galois group  $G$ . Hence  $B * G \cong \text{Hom}_{B^G}(B, B)$  ([2], Theorem 1). By (2),  $B$  is centrally projective over  $B^G$ . Thus,  $B * G$  ( $\cong \text{Hom}_{B^G}(B, B)$ ) is  $H$ -separable over  $B$  ([6], Proposition 11).

(4) We first claim that  $V_{B * G}(C) = B$ . Clearly,  $B \subset V_{B * G}(C)$ . Let  $\sum_{j=1}^n b_j g_j$  in  $V_{B * G}(C)$  for some  $b_j \in B$ . Then  $c(\sum_{j=1}^n b_j g_j) = (\sum_{j=1}^n b_j g_j)c$  for each  $c$  in  $C$ , so  $cb_j = b_j g_j(c)$ , that is,  $b_j(c - g_j(c)) = 0$  for each  $g_j \in G$  and  $c \in C$ . Since  $B = BJ_j^{(C)}$  for each  $g_j \neq 1$ , there exist  $b_i^{(j)} \in B$  and  $c_i^{(j)} \in C, i = 1, 2, \dots, m$  such that  $\sum_{i=1}^m b_i^{(j)}(c_i^{(j)} - g_j(c_i^{(j)})) = 1$ . Hence  $b_j = \sum_{i=1}^m b_i^{(j)}(c_i^{(j)} - g_j(c_i^{(j)}))b_j = \sum_{i=1}^m b_i^{(j)}b_j(c_i^{(j)} - g_j(c_i^{(j)})) = 0$  for each  $g_j \neq 1$ . This implies that  $\sum_{j=1}^n b_j g_j = b_1 \in B$ . Hence  $V_{B * G}(C) \subseteq B$ , and so  $V_{B * G}(C) = B$ . Therefore,  $V_{B * G}(B) \subset V_{B * G}(C) = B$ . Thus  $V_{B * G}(B) = V_B(B) = C$ .

We now show some characterizations of a center Galois extension  $B$ .

**THEOREM 3.2.** *The following statements are equivalent.*

- (1)  $B$  is a center Galois extension of  $B^G$ .
- (2)  $B = BJ_j^{(C)}$  for each  $g_j \neq 1$  in  $G$ .
- (3)  $B$  is a Galois extension of  $B^G$  with a Galois system  $\{b_i \in B, c_i \in C, i = 1, 2, \dots, m\}$  for some integer  $m$ .

(4)  $B$  is a Galois central extension of  $B^G$ .

(5)  $B^G C = B^G C J_j^{(C)}$  for each  $g_j \neq 1$  in  $G$ .

PROOF. (1)  $\implies$  (2) By hypothesis,  $C$  is a Galois extension of  $C^G$  with Galois group  $G|_C \cong G$ . Hence  $C = C J_j^{(C)}$  for each  $g_j \neq 1$  in  $G$  ([3], Proposition 1.2, p.80). Thus,  $B = B J_j^{(C)}$  for each  $g_j \neq 1$  in  $G$ .

(2)  $\implies$  (1) Since  $B = B J_j^{(C)}$  for each  $g_j \neq 1$  in  $G$ ,  $B * G$  is  $H$ -separable over  $B$  by Lemma 3.1-(3) and  $V_{B * G}(B) = C$  by Lemma 3.1-(4). Thus,  $C$  is a Galois extension of  $C^G$  with Galois group  $G|_C \cong G$  by ([1], Proposition 4).

(1)  $\implies$  (3) This is Lemma 3.1-(1).

(3)  $\implies$  (1) Since  $B$  is a Galois extension of  $B^G$  with a Galois system  $\{b_i \in B, c_i \in C, i = 1, 2, \dots, m\}$  for some integer  $m$ , we have  $\sum_{i=1}^m b_i g_j(c_i) = \delta_{1,j}$ . Hence  $\sum_{i=1}^m b_i (c_i - g_j(c_i)) = 1$  for each  $g_j \neq 1$  in  $G$ . So for every  $b \in B$ ,  $b = \sum_{i=1}^m b b_i (c_i - g_j(c_i)) \in B J_j^{(C)}$ . Therefore,  $B = B J_j^{(C)}$  for each  $g_j \neq 1$  in  $G$ . Thus,  $B$  is a center Galois extension of  $B^G$  by (2)  $\implies$  (1).

(1)  $\implies$  (4) Since  $C$  is a Galois algebra with Galois group  $G|_C \cong G$ ,  $B$  and  $B^G C$  are Galois extensions of  $B^G$  with Galois group  $G|_{B^G C} \cong G$ . Noting that  $B^G C \subset B$ , we have  $B = B^G C$ , that is,  $B$  is a central extension of  $B^G$ . But  $B$  is a Galois extension of  $B^G$ , so  $B$  is a Galois central extension of  $B^G$ .

(4)  $\implies$  (1) By hypothesis,  $B = B^G C$  is a Galois extension of  $B^G$ . Hence there exists a Galois system  $\{a_i; b_i \in B, i = 1, 2, \dots, m\}$  for some integer  $m$  such that  $\sum_{i=1}^m a_i g_j(b_i) = \delta_{1,j}$ . But  $B = B^G C$ , so  $a_i = \sum_{k=1}^{n_{a_i}} b_k^{(a_i)} c_k^{(a_i)}$  and  $b_i = \sum_{l=1}^{n_{b_i}} b_l^{(b_i)} c_l^{(b_i)}$  for some  $a_k^{(a_i)}, b_l^{(b_i)}$  in  $B^G$  and  $c_k^{(a_i)}, c_l^{(b_i)}$  in  $C$ ,  $k = 1, 2, \dots, n_{a_i}, l = 1, 2, \dots, n_{b_i}, i = 1, 2, \dots, m$ . Therefore,

$$\begin{aligned} \delta_{1,j} &= \sum_{i=1}^m a_i g_j(b_i) = \sum_{i=1}^m \sum_{k=1}^{n_{a_i}} b_k^{(a_i)} c_k^{(a_i)} g_j\left(\sum_{l=1}^{n_{b_i}} b_l^{(b_i)} c_l^{(b_i)}\right) \\ &= \sum_{i=1}^m \sum_{k=1}^{n_{a_i}} b_k^{(a_i)} c_k^{(a_i)} \sum_{l=1}^{n_{b_i}} b_l^{(b_i)} g_j(c_l^{(b_i)}) = \sum_{i=1}^m \sum_{k=1}^{n_{a_i}} \sum_{l=1}^{n_{b_i}} (b_k^{(a_i)} c_k^{(a_i)} b_l^{(b_i)}) g_j(c_l^{(b_i)}). \end{aligned}$$

This shows that  $\{b_{k,l}^{(a_i,b_i)} = b_k^{(a_i)} c_k^{(a_i)} b_l^{(b_i)} \in B; c_{k,l}^{(a_i,b_i)} = c_l^{(b_i)} \in C, k = 1, 2, \dots, n_{a_i}, l = 1, 2, \dots, n_{b_i}, i = 1, 2, \dots, m\}$  is a Galois system for  $B$ . Thus,  $B$  is a center Galois extension of  $B^G$  by (3)  $\implies$  (1).

(1)  $\implies$  (5) Since  $B$  is a center Galois extension of  $B^G$ ,  $B = B J_j^{(C)}$  for each  $g_j \neq 1$  in  $G$  by (1)  $\implies$  (2) and  $B = B^G C$  by (1)  $\implies$  (4). Thus,  $B^G C = B^G C J_j^{(C)}$  for each  $g_j \neq 1$  in  $G$ .

(5)  $\implies$  (1) Since  $B^G C = B^G C J_j^{(C)}$  for each  $g_j \neq 1$  in  $G$ ,  $B = B J_j^{(C)}$  for each  $g_j \neq 1$  in  $G$ . Thus,  $B$  is a center Galois extension of  $B^G$  by (2)  $\implies$  (1).

The characterization of a commutative Galois extension  $C$  in terms of the ideals generated by  $\{c - g(c) \mid c \in C\}$  for  $g \neq 1$  in  $G$  is an immediate consequence of Theorem 3.2.

**COROLLARY 3.3.** *A commutative ring  $C$  is a Galois extension of  $C^G$  if and only if  $C = C J_j^{(C)}$ , the ideal generated by  $\{c - g_j(c) \mid c \in C\}$  is  $C$  for each  $g_j \neq 1$  in  $G$ .*

PROOF. Let  $B = C$  in Theorem 3.2. Then, the corollary is an immediate consequence of Theorem 3.2-(2).

By Theorem 3.2, we derive several characterizations of a Galois central extension  $B$ .

**COROLLARY 3.4.** *If  $B$  is a central extension of  $B^G$  (that is,  $B = B^G C$ ), then the following statements are equivalent.*

- (1)  $B$  is a Galois extension of  $B^G$ .
- (2)  $B$  is a center Galois extension of  $B^G$ .
- (3)  $B * G$  is  $H$ -separable over  $B$ .
- (4)  $B = C J_j^{(B)}$  for each  $g_j \neq 1$  in  $G$ .
- (5)  $B = B J_j^{(B)}$  for each  $g_j \neq 1$  in  $G$ .

PROOF. (1)  $\iff$  (2) This is given by (1)  $\iff$  (4) in Theorem 3.2.

(2)  $\implies$  (3) This is Lemma 3.1-(3).

(3)  $\implies$  (1) Since  $B * G$  is  $H$ -separable over  $B$ ,  $B$  is a Galois extension of  $B^G$  ([1], Proposition 2).

Since  $B = B^G C$  by hypothesis, it is easy to see that  $J_j^{(B)} = B^G J_j^{(C)}$  for each  $g_j$  in  $G$ . Thus,  $B = C J_j^{(B)}$ ,  $B = B J_j^{(B)}$ , and  $B = B J_j^{(C)}$  are equivalent. This implies that (2)  $\iff$  (4)  $\iff$  (5) by Theorem 3.2-(2).

We call a ring  $B$  the DeMeyer-Kanzaki Galois extension of  $B^G$  if  $B$  is an Azumaya  $C$ -algebra and  $B$  is a center Galois extension of  $B^G$  (for more about the DeMeyer-Kanzaki

Galois extensions, see [2]). Clearly, the class of center Galois extensions is broader than the class of the DeMeyer-Kanzaki Galois extensions. We conclude the present paper with two examples. (1) the DeMeyer-Kanzaki Galois extension of  $B^G$  and (2) a center Galois extension of  $B^G$ , but not the DeMeyer-Kanzaki Galois extension of  $B^G$ .

**EXAMPLE 1.** Let  $C$  be the field of complex numbers, that is,  $C = R + R\sqrt{-1}$  where  $R$  is the field of real numbers,  $B = C[i, j, k]$  the quaternion algebra over  $C$ , and  $G = \{1, g \mid g(c_1 + c_i i + c_j j + c_k k) = g(c_1) + g(c_i)i + g(c_j)j + g(c_k)k \text{ for each } b = c_1 + c_i i + c_j j + c_k k \in C[i, j, k] \text{ and } g(u + v\sqrt{-1}) = u - v\sqrt{-1} \text{ for each } c = u + v\sqrt{-1} \in C\}$ . Then

- (1) The center of  $B$  is  $C$ .
- (2)  $B$  is an Azumaya  $C$ -algebra.
- (3)  $C$  is a Galois extension of  $C^G$  with Galois group  $G|_C \cong G$  and a Galois system  $\{a_1 = \frac{1}{\sqrt{2}}, a_2 = \frac{1}{\sqrt{2}}\sqrt{-1}; b_1 = \frac{1}{\sqrt{2}}, b_2 = -\frac{1}{\sqrt{2}}\sqrt{-1}\}$ .
- (4)  $B$  is the DeMeyer-Kanzaki Galois extension of  $B^G$  by (2) and (3).
- (5)  $B^G = R[i, j, k]$ .
- (6)  $B = B^G C$ , so  $B$  is a central extension of  $B^G$ .
- (7)  $J_g^{(C)} = R\sqrt{-1}$ .
- (8)  $B = B J_g^{(C)}$  since  $1 = -\sqrt{-1}\sqrt{-1} \in B J_g^{(C)}$ .
- (9)  $J_g^{(B)} = R\sqrt{-1} + R\sqrt{-1}i + R\sqrt{-1}j + R\sqrt{-1}k$ .
- (10)  $B = C J_g^{(B)}$ .

**EXAMPLE 2.** By replacing in Example 1 the field of complex numbers  $C$  with the ring  $C = Z \oplus Z$  where  $Z$  is the ring of integers,  $g(a, b) = (b, a)$  for all  $(a, b) \in C$ , and  $G = \{1, g \mid g(c_1 + c_i i + c_j j + c_k k) = g(c_1) + g(c_i)i + g(c_j)j + g(c_k)k \text{ for each } b = c_1 + c_i i + c_j j + c_k k \in B = C[i, j, k]\}$ . Then

- (1) The center of  $B$  is  $C$ .
- (2)  $C$  is a Galois extension of  $C^G$  with Galois group  $G|_C \cong G$  and a Galois system  $\{a_1 = (1, 0), a_2 = (0, 1); b_1 = (1, 0), b_2 = (0, 1)\}$ .
- (3)  $B$  is not an Azumaya  $C$ -algebra (for  $\frac{1}{2} \notin C$ ), and so  $B$  is not the DeMeyer-Kanzaki Galois extension of  $B^G$ .
- (4)  $C^G = \{(a, a) \mid a \in Z\} \cong Z$
- (5)  $B^G = C^G[i, j, k]$ .

- (6)  $B = B^G C$ , so  $B$  is a central extension of  $B^G$ .
- (7)  $J_g^{(C)} = \{(a, -a) \mid a \in Z\} = Z(1, -1)$ .
- (8)  $B = B J_g^{(C)}$  since  $1 = (1, 1) = (1, -1)(1, -1) \in B J_g^{(C)}$ .
- (9)  $J_g^{(B)} = Z(1, -1) + Z(1, -1)i + Z(1, -1)j + Z(1, -1)k$ .
- (10)  $B = C J_g^{(B)}$ .

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