On the Cartan Lie algebras $H(\mathcal{F})$ of characteristic 2

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Abstract

Let k be a field of characteristic $p = 2$, E an n-dimensional vector space over k, N the set of natural numbers, and $O(E)$ an algebra over k generated by $\{x^{(h)} \,|\, x \in E, h \in N\}$ where $\{x^{(h)}\}$ satisfy the following:

$$\begin{aligned} x^{(0)} &= 1 \\ (x+y)^{(h)} &= \sum_{i=0}^{h} x^{(i)} y^{(h-i)} \\ x^{(h)} x^{(i)} &= \binom{h+i}{h} x^{(h+i)} \\ (\alpha x)^{(h)} &= \alpha^{(h)} x^{(h)} \end{aligned}$$

for all $x, y \in E$, $\alpha \in k$, and $h, i \in N$. Given a flag $\mathcal{F} : E = E_0 \supseteq E_1 \supseteq \cdots \supseteq E_t \neq \{0\}$, $O(\mathcal{F})$ is a subalgebra of $O(E)$ generated by $\{x^{(2^i)} | \, x \in E_i, i = 0, 1, \cdots, t\}$. We can choose a basis $x_1, x_2, \cdots, x_n$ of E such that

$$\{x_1^{(h_1)}, x_2^{(h_2)}, \cdots, x_n^{(h_n)} \,|\, 0 \leq h_i < 2^{m_i} \text{ for some } 1 \leq m_1 \leq m_2 \leq \cdots \leq m_n\}$$

is a basis for $O(\mathcal{F})$. Hence $\text{Dim}(O(\mathcal{F})) = 2^m$ where $m = \sum_{i=0}^{n} m_i$. A derivative D of $O(\mathcal{F})$ is called a special derivative if $D(x^{(h)}) = x^{(h-1)} D(x)$ for $x \in E_i$, $h < 2^{i+1}$, $i = 0, 1, \cdots, t$. The set of special derivatives of $O(\mathcal{F})$ forms a Lie algebra which is a Cartan type Lie algebra and is denoted by $W(\mathcal{F})$.

Let E^* be the dual space of E and $\xi_1, \xi_2, \cdots, \xi_n$ in E^* be a dual basis of $x_1, x_2, \cdots, x_n$. We define a derivative ∂_i in $W(\mathcal{F})$, for $i = 0, 1, \cdots, n$, by

$$\partial_i(x^{(h)}) = \begin{cases} 0, & h = 0 \\ x^{(h-1)} \xi_i(x), & h > 0 \end{cases} \quad x \in E_j, \; h < 2^{i+1}, \; i = 1, 2, \cdots, n.$$

Then

$$W(\mathcal{F})=\sum_{i=1}^{n} O(\mathcal{F})\partial_i.$$

Assume that $n=2n'$. We define

$$\widetilde{H}(\mathcal{F})=\{D\in W(\mathcal{F})|\, D\omega=0\} \text{ and}$$

$$H(\mathcal{F})=[[\widetilde{H}(\mathcal{F}),\widetilde{H}(\mathcal{F})],[\widetilde{H}(\mathcal{F}),\widetilde{H}(\mathcal{F})]]$$

where $\omega=\sum_{i=1}^{n}\sum_{j=1}^{n}\omega_{ij}dx_i\wedge dx_j$, $(\omega_{ij})_{n\times n}=\Phi\begin{pmatrix}0 & I_{n'}\\ 0 & 0\end{pmatrix}\Phi^T$, Φ is an invertible $n\times n$ matrix, and Φ^T is the transpose of Φ. $\widetilde{H}(\mathcal{F})$ and $H(\mathcal{F})$ are called Hamiltonian Lie algebras. When $m=n$ or $m_n=1$, $H(\mathcal{F})$ is denoted by $H_{n'}$. A. I. Kostrikin and I. R. Safarevic studied $\widetilde{H}(\mathcal{F})$ and $H(\mathcal{F})$ over a field of characteristic $p\geq 3$. In this paper, we investigate $\widetilde{H}(\mathcal{F})$ and $H(\mathcal{F})$ when $p=2$ and shall prove the following:

(1) $\mathrm{Dim}(\widetilde{H}(\mathcal{F}))=2^m+n-1$ and $\mathrm{Dim}(H(\mathcal{F}))=2^m-2$.

(2) $H(\mathcal{F})\cong O'(\mathcal{F})/k$.

(3) For the gradation $H(\mathcal{F})=L_{-1}+L_0+L_1+\cdots+L_r$ induced by $W(\mathcal{F})_i=\sum_{j=1}^{n}O(\mathcal{F})_{i+1}\partial_j$, L_{-1} is an irreducible L_0-module when $n>3$ or $n=2$ and $m_1>1$, and L_{-1} is a reducible L_0-module when $n=2$ and $m_1=1$.

(4) $H(\mathcal{F})$ is a simple Lie algebra when $n>3$ or $n=2$ and $m_1>1$, and $H(\mathcal{F})$ is not a simple Lie algebra when $n=2$ and $m_1=1$, in particular, H_1 is not a simple Lie algebra.

(5) $H(\mathcal{F})$ is a Lie p-algebra if and only if $m=n$.

(6) $H_{n'}=\langle x\,|\, x\in H_{n'},(adx)^2=0\rangle$.