On a Class of Azumaya Galois Extensions

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Abstract. Let B be a ring with 1, G a finite automorphism group of B, C the center of B, $K = \{g \in G \mid g(c) = c$ for all $c \in C\}$, and $J_g = \{b \in B \mid bx = g(x)b$ for all $x \in B\}$ for each $g \in G$. Then, B is an Azumaya Galois extension with Galois group G and $J_g = \{0\}$ for each $g \notin K$ if and only if B is a commutator Galois extension of B^K with Galois group K and B^K is a DeMeyer-Kanzaki Galois extension of B^G with Galois group G/K. More equivalent conditions are also given in terms of Azumaya skew group rings.

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1 Introduction

Let B be a Galois algebra with Galois group G, C the center of B, and $K = \{g \in G \mid g(c) = c$ for all $c \in C\}$. In [3], it was shown that if C is indecomposable (that is, C contains no idempotents but 0 and 1), then B is a central Galois algebra with Galois group K and C is a commutative Galois algebra over C^G with Galois group G/K. In [5], the result was generalized to a Galois algebra B with nontrivial idempotents: B is a central Galois algebra with Galois group K and C is a commutative Galois algebra with Galois group G/K if and only if $J_g = \{0\}$ for each $g \notin K$ where $J_g = \{b \in B \mid bx = g(x)b$ for all $x \in B\}$ for an $g \in G$. The purpose of the present paper is to give a further generalization to the class of Azumaya Galois extensions for which $J_g = \{0\}$ for each $g \notin K$ where an Azumaya Galois extension B with Galois group G is a Galois extension of B^G which is an Azumaya C^G-algebra (for more about an Azumaya Galois extension, see [2] and [6]). We recall that a Galois extension B with Galois group G is a commutator Galois extension if the commutator subring of B^G in B, $V_B(B^G)$, is a Galois extension with Galois group induced by and isomorphic with G ([8]), and a DeMeyer-Kanzaki Galois extension if B is an Azumaya algebra over C which is a Galois algebra over C^G with Galois group induced by and isomorphic with G ([3],[5]). We shall show the following equivalent conditions:

(1) B is an Azumaya Galois extension of B^G with Galois group G and $J_g = \{0\}$ for each $g \notin K$.

(2) B is a commutator Galois extension of B^K with Galois group K and B^K is a DeMeyer-Kanzaki Galois extension of B^G with Galois group G/K.

(3) $B \cong B^G C \otimes_C V_B(B^G)$ such that $B^G C$ is a DeMeyer-Kanzaki Galois extension of B^G with Galois group G/K and $V_B(B^G)$ is a central Galois algebra over C with Galois group K.

(4) The skew group rings $B * G$ and $(B^G C) * (G/K)$ are Azumaya C^G-algebras.

Similar results are also obtained for commutator Galois extensions.

2 Basic Definitions and Notations

Throughout, we assume that B is a ring with 1, C the center of B, G a finite automorphism group of B, $K = \{g \in G \,|\, g(c) = c$ for all $c \in C\}$, and $J_g = \{b \in B \,|\, bx = g(x)b$ for all $x \in B\}$ for each $g \in G$. We denote B^G the set $\{a \in B \,|\, g(a) = a$ for all $g \in G\}$. For a subring A of B, we denote the commutator subring of A in B by $V_B(A)$ and $\{a \in A \,|\, ax = g(x)a$ for all $x \in A\}$ by $J_g^{(A)}$ for each $g \in G$.

As given in [7], B is called a Galois extension of B^G with Galois group G if there exist elements $\{a_i, b_i$ in B, $i = 1, 2, ..., m$ for some integer $m\}$ such that $\sum_{i=1}^m a_i g(b_i) = \delta_{1,g}$, and such a set $\{a_i, b_i\}$ is called a G-Galois system for B. A Galois extension B with Galois group G is called an Azumaya Galois extension if B^G is an Azumaya algebra over C^G ([2],[6]), and a DeMeyer-Kanzaki Galois extension if B is an Azumaya algebra over C

which is a Galois algebra over C^G with Galois group induced by and isomorphic with G ([3],[5]). A Galois extension B with Galois group G is called a commutator Galois extension of B^G if the commutator subring of B^G in B, $V_B(B^G)$, is a Galois extension with Galois group induced by and isomorphic with G ([8]). We note that the class of DeMeyer-Kanzaki Galois extensions is contained in the class of Azumaya Galois extensions which is a subclass of commutator Galois extensions. Throughout this paper, for an G-invariant subring A of B, that A is a Galois extension of A^G with Galois group G means that A is a Galois extension of A^G with Galois group induced by and isomorphic with G.

3. Characterizations

In this section, keeping all notations of section 2 we shall give several characterizations of the class of Azumaya Galois extensions for which $J_g = \{0\}$ for each $g \notin K$. We begin with an expression of an Azumaya Galois extension, a DeMeyer-Kanzaki Galois extension, and a commutator Galois extension, respectively.

Lemma 3.1

If B is a DeMeyer-Kanzaki Galois extension with Galois group G, then $B = B^G C \cong B^G \otimes_{C^G} C$ where B^G is an Azumaya C^G-algebra and C is a commutative Galois algebra with Galois group G.

Proof. It is given in Lemma 2 in [3].

Lemma 3.2

If B is a commutator Galois extension of B^G with Galois group G, then
$B = B^G \cdot V_B(B^G)$ such that $V_B(B^G)$ is a Galois algebra over C^G with Galois group G.

Proof. Since B is a commutator Galois extension of B^G with Galois group G, $V_B(B^G)$ is a Galois algebra over C^G with Galois group G by Lemma 3.1 in [8]. But then B and

$B^G \cdot V_B(B^G)$ are Galois extensions of B^G with Galois group G such that $B^G \cdot V_B(B^G) \subset B$; and so $B = B^G \cdot V_B(B^G)$.

Lemma 3.3

If B is an Azumaya Galois extension with Galois group G, then $B = B^G \cdot V_B(B^G) \cong B^G \otimes_{C^G} V_B(B^G)$ where B^G is an Azumaya C^G-algebra and $V_B(B^G)$ is a Galois algebra over C^G with Galois group G.

Proof. Since B is an Azumaya Galois extension with Galois group G, B^G is an Azumaya C^G-algebra and $V_B(B^G)$ is a Galois algebra over C^G with Galois group G ([1], Theorem 2). Hence B is a commutator Galois extension of B^G with Galois group G. Thus by Lemma 3.2, $B = B^G \cdot V_B(B^G) \cong B^G \otimes_{C^G} V_B(B^G)$ ([1], Theorem 2).

We need one more lemma on patching two different Galois systems to one.

Lemma 3.4

If B is a Galois extension with Galois group K and C is a Galois algebra with Galois group G/K, then B is a Galois extension with Galois group G.

Proof. Let $\{a_i, b_i$ in B, $i = 1, 2, ..., m$ for some integer $m\}$ be a K-Galois system for B and $\{c_j, d_j$ in C, $j = 1, 2, ..., k$ for some integer $k\}$ be a G/K-Galois system for C. Then $\sum_{i=1}^m a_i b_i = 1$, $\sum_{i=1}^m a_i g(b_i) = 0$ for $g \neq 1$ in K, $\sum_{j=1}^k c_j d_j = 1$, and $\sum_{j=1}^k c_j g(d_j) = 0$ for $g \notin K$. Hence

$$\sum_{i=1}^m \sum_{j=1}^k (a_i c_j)(b_i d_j) = (\sum_{i=1}^m a_i b_i)(\sum_{j=1}^k c_j d_j) = 1 \text{ and}$$

$$\sum_{i=1}^m \sum_{j=1}^k (a_i c_j) g(b_i d_j) = \big(\sum_{i=1}^m a_i g(b_i)\big)\big(\sum_{j=1}^k c_j g(d_j)\big) = 0 \text{ for } g \neq 1 \text{ in } G.$$

Thus $\{a_i c_j, b_i d_j$ in B, $i = 1, 2, ..., m$ and $j = 1, 2, ..., k\}$ is a G-Galois system for B. Therefore B is a Galois extension with Galois group G.

Theorem 3.5

The following statements are equivalent:

(1) B is an Azumaya Galois extension of B^G with Galois group G and $J_g = \{0\}$ for each $g \notin K$.

(2) B is a commutator Galois extension of B^K with Galois group K and B^K is a DeMeyer-Kanzaki Galois extension of B^G with Galois group G/K.

(3) $B \cong B^G C \otimes_C V_B(B^G)$ such that $B^G C$ is a DeMeyer-Kanzaki Galois extension of B^G with Galois group G/K and $V_B(B^G)$ is a central Galois algebra over C with Galois group K.

(4) The skew group rings $B * G$ and $(B^G C) * (G/K)$ are Azumaya C^G-algebras.

Proof. (1) $\Longrightarrow$ (2) Since B is an Azumaya Galois extension with Galois group G, $B = B^G \cdot V_B(B^G) \cong B^G \otimes_{C^G} V_B(B^G)$ such that $V_B(B^G)$ is a Galois algebra over C^G with Galois group G by Lemma 3.3. From the expression $B = B^G \cdot V_B(B^G) \cong B^G \otimes_{C^G} V_B(B^G)$, we can see that the center of $V_B(B^G)$ is C and $J_g = J_g^{(V_B(B^G))}$ for each $g \in G$, so that $J_g = \{0\}$ for each $g \notin K$ implies that $J_g^{(V_B(B^G))} = \{0\}$ for each $g \notin K$. Hence $V_B(B^G)$ is a central Galois algebra over C with Galois group K and C is a commutative Galois algebra with Galois group G/K ([5], Proposition 3). This implies that both $B^G C$ and B^K are Galois extensions of B^G with Galois group G/K (for $C \subset B^K$). Therefore, noting that $B^G C \subset B^K$, we have that $B^K = B^G C$ which is a DeMeyer-Kanzaki Galois extension with Galois group G/K, and then $V_B(B^K) = V_B(B^G C) = V_B(B^G)$ which was already shown to be a central Galois algebra with Galois group K, that is, B is a commutator Galois extension of B^K with Galois group K. Thus (2) holds.

(2) $\Longrightarrow$ (1) Since B is a commutator Galois extension of B^K with Galois group K, $B = B^K \cdot V_B(B^K)$ by Lemma 3.2 such that $V_B(B^K)$ is a Galois algebra over C^K with Galois group K and has center C. By hypothesis, B^K is a DeMeyer-Kanzaki Galois extension of B^G with Galois group G/K, so $B^K = B^G Z \cong B^G \otimes_{Z^G} Z$ where Z is the center of B^K

by Lemma 3.1, B^G $(= (B^K)^{G/K})$ is an Azumaya Z^G-algebra, and Z is a commutative Galois algebra over Z^G with Galois group G/K. Noting that $B = B^K \cdot V_B(B^K)$ and Z is the center of B^K, we have that Z is contained in the center of B, that is, $Z \subset C$. But $C \subset B^K$, so $C \subset Z$. Thus $Z = C$; and so B^G is an Azumaya C^G-algebra. Thus to show that B is an Azumaya Galois extension of B^G with Galois group G, it suffices to show that B is a Galois extension of B^G with Galois group G. In fact, since B is a Galois extension of B^K with Galois group K and C $(= Z)$ is a Galois algebra with Galois group G/K, B is a Galois extension of B^G with Galois group G by Lemma 3.4.

Next we claim that $J_g = \{0\}$ for each $g \notin K$. Since $V_B(B^K)$ is a Galois algebra over C^K with Galois group K, $V_B(B^K) = \oplus \sum_{g\in K} J_g^{(V_B(B^K))}$ ([5], Theorem 1). But B is a Galois extension of B^G with Galois group G, so $V_B(B^G) = \oplus \sum_{g\in G} J_g$ ([5], Proposition 1). Noting that $B^K = B^G C$ and $J_g^{(V_B(B^K))} = J_g$ for each $g \in K$, we have that $\oplus \sum_{g\in G} J_g = V_B(B^G) = V_B(B^G C) = V_B(B^K) = \oplus \sum_{g\in K} J_g^{(V_B(B^K))} = \oplus \sum_{g\in K} J_g$. Thus $J_g = \{0\}$ for each $g \notin K$.

(1) $\Longrightarrow$ (3) By Lemma 3.3, $B = B^G \cdot V_B(B^G) \cong B^G \otimes_{C^G} V_B(B^G) \cong B^G \otimes_{C^G} C \otimes_C V_B(B^G) \cong B^G C \otimes_C V_B(B^G)$. By the proof of (1) $\Longrightarrow$ (2), $B^G C = B^K$ which is a DeMeyer-Kanzaki Galois extension of B^G with Galois group G/K and $V_B(B^G)$ is a central Galois algebra over C with Galois group K, so statement (3) holds.

(3) $\Longrightarrow$ (1) Since $B \cong B^G C \otimes_C V_B(B^G)$, C is the center of $B^G C$. Hence C is a Galois algebra with Galois group G/K for $B^G C$ is a DeMeyer-Kanzaki Galois extension of B^G with Galois group G/K. By hypothesis, $V_B(B^G)$ is a central Galois algebra over C with Galois group K, so B is a Galois extension with Galois group K. Hence by Lemma 3.4, B is a Galois extension with Galois group G. Moreover, since $B^G C$ is a DeMeyer-Kanzaki Galois extension of B^G with Galois group G/K, B^G is an Azumaya algebra over C^G by Lemma 3.1. Thus B is an Azumaya Galois extension of B^G with Galois group G.

Next we show that $J_g = \{0\}$ for each $g \notin K$. Since B is a Galois extension of B^G with

Galois group G, $V_B(B^G) = \oplus\sum_{g\in G} J_g$ ([5], Proposition 1). By hypothesis, $V_B(B^G)$ is a central Galois algebra over C with Galois group K, so $V_B(B^G) = \oplus\sum_{g\in K} J_g^{(V_B(B^G))}$ ([5], Theorem 1). Since B^GC and $V_B(B^G)$ are Azumaya C-algebras, that $B \cong B^GC\otimes_C V_B(B^G)$ implies that $B = B^GC\cdot V_B(B^G) = B^G\cdot V_B(B^G)$ by the multiplication map ([4], Theorem 4.4, page 58). This implies that $J_g^{(V_B(B^G))} = J_g$ for each $g \in G$. Hence $\oplus\sum_{g\in G} J_g = V_B(B^G) = \oplus\sum_{g\in K} J_g^{(V_B(B^G))} = \oplus\sum_{g\in K} J_g$. Thus $J_g = \{0\}$ for each $g \notin K$.

(1) $\Longrightarrow$ (4) Since B is an Azumaya Galois extension of B^G with Galois group G, $B*G$ is an Azumaya C^G-algebra ([2], Theorem 1). Moreover, by (1) $\Longrightarrow$ (3), B^GC is a DeMeyer-Kanzaki Galois extension of B^G with Galois group G/K, so B^GC is an Azumaya Galois extension of B^G with Galois group G/K. Thus, by Theorem 1 in [2] again, $(B^GC)*(G/K)$ is an Azumaya C^G-algebra.

(4) $\Longrightarrow$ (2) Since $B*G$ is an Azumaya C^G-algebra, B is an Azumaya Galois extension of B^G with Galois group G ([2], Theorem 1). Hence, by Lemma 3.3, $B = B^G\cdot V_B(B^G) \cong B^G\otimes_{C^G} V_B(B^G) \cong B^G\otimes_{C^G} C\otimes_C V_B(B^G) \cong B^GC\otimes_C V_B(B^G)$. Thus both B^GC and $V_B(B^G)$ have center C. Therefore $V_{B^GC}(B^G) = V_{B^GC}(B^GC) = C$. Moreover, by hypothesis, $(B^GC)*(G/K)$ is an Azumaya C^G-algebra, so B^GC is an Azumaya Galois extension of B^G with Galois group G/K by Theorem 1 in [2] again. Hence, by Lemma 3.3, C $(= V_{B^GC}(B^G))$ is a Galois algebra with Galois group G/K. This implies that $B^GC = B^K$. Thus B^K is a DeMeyer-Kanzaki Galois extension of B^G with Galois group G/K.

Next we claim that B is a commutator Galois extension of B^K with Galois group K, that is, $V_B(B^K)$ is a Galois extension with Galois group K. In fact, since B is an Azumaya Galois extension of B^G with Galois group G, by Lemma 3.3, $V_B(B^G)$ is a Galois algebra over C^G with Galois group G. Hence $V_B(B^G)$ is a Galois extension with Galois group K. But $V_B(B^G) = V_B(B^GC) = V_B(B^K)$, so $V_B(B^K)$ is a Galois extension with Galois group K. This completes the proof.

By using the similar argument as given in Theorem 3.5, we can show two equivalent conditions for a commutator Galois extension for which $J_g = \{0\}$ for each $g \notin K$. Recall that B is called a center Galois extension of B^G with Galois group G if C is a Galois algebra over C^G with Galois group G ([7]).

Theorem 3.6

The following statements are equivalent:

(1) B is a commutator Galois extension with Galois group G for which $J_g = \{0\}$ for each $g \notin K$.

(2) $B = B^G C \cdot V_B(B^G)$ such that $B^G C$ is a center Galois extension of B^G with Galois group G/K and $V_B(B^G)$ is a central Galois algebra over C with Galois group K.

(3) B is a commutator Galois extension of B^K with Galois group K and B^K is a center Galois extension of B^G with Galois group G/K.

Remark.

The equivalent conditions of (1), (2), and (3) of the Theorem 3.5 generalize the Kanzaki's result ([5], Proposition 3) to Azumaya Galois extensions, and Theorem 3.6 is a further generalization of Theorem 3.5 to commutator Galois extensions.

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