

On Center Hopf Galois Extensions

George Szeto and Lianyong Xue

Department of Mathematics, Bradley University
Peoria, Illinois 61625, USA

Email: szeto@hilltop.bradley.edu and lxue@hilltop.bradley.edu

ABSTRACT. Let H be a finite dimensional Hopf algebra over a field k , H^* the dual Hopf algebra of H , and B a left H -module algebra with center C . Then B is called a right center H^* -Galois extension of B^H if C is a right H^* -Galois and separable extension of C^H . Such a B is characterized in terms of the smash product $B\#H$ and some properties of $B\#H$ are obtained.

1. Introduction

In [11], the class of center Galois extensions of rings were studied, where B is called a center Galois extension of B^G with a finite Galois group G if its center C is a Galois extension of C^G with Galois group induced by and isomorphic with G . A center Galois extension B of B^H becomes a DeMeyer-Kanzaki Galois extension if B^G is an Azumaya algebra over C^G as given in [5]. A Galois extension B with Galois group G such that B^G is an Azumaya algebra over C^G is called an Azumaya Galois extension with Galois group G ([1,2]). Recently, the class of Azumaya Galois extensions were generalized to the class of Hopf Azumaya Galois extensions ([9,12]). The purpose of the present paper is to generalize the class of center Galois extensions to the class of center Hopf Galois extensions. Let H be a finite dimensional Hopf algebra over a field k , H^* the dual Hopf algebra of H , and B a left H -module algebra with center C . We call B a right center H^* -Galois extension

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of B^H if C is a right H^* -Galois and separable extension of C^H . We shall characterize a right center H^* -Galois extension B of B^H in terms of the Hirata separable smash product $B\#H$ and the Azumaya smash product $V_B(B^H)\#H$ respectively where $V_B(B^H)$ is the commutator subring of B^H in B . Consequently, some properties of $B\#H$ are derived by using the Ulbrich results in [15].

2. Basic definitions and notations

Throughout, H denotes a finite dimensional Hopf algebra over a field k with comultiplication Δ and counit ε , H^* the dual Hopf algebra of H , B a left H -module algebra, C the center of B , $B^H = \{b \in B \mid hb = \varepsilon(h)b \text{ for all } h \in H\}$ which is called the H -invariants of B , and $B\#H$ the smash product of B with H where $B\#H = B \otimes_k H$ such that for all $b\#h$ and $b'\#h'$ in $B\#H$, $(b\#h)(b'\#h') = \sum b(h_1b')\#h_2h'$ where $\Delta(h) = \sum h_1 \otimes h_2$. B is called a right H^* -Galois extension of B^H if B is a right H^* -comodule algebra with structure map $\rho : B \rightarrow B \otimes_k H^*$ such that $\beta : B \otimes_{B^H} B \rightarrow B \otimes_k H^*$ is a bijection where $\beta(a \otimes b) = (a \otimes 1)\rho(b)$.

For a subring A of B with the same identity 1, we denote the commutator subring of A in B by $V_B(A)$. As given in [4], we call B a separable extension of A if there exist $\{a_i, b_i$ in B , $i = 1, 2, \dots, m$ for some integer $m\}$ such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where \otimes is over A . An Azumaya algebra is a separable extension of its center. A ring B is called a Hirata separable extension of A if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B -bimodule. A right H^* -Galois extension B is called a center right H^* -Galois extension of B^H if C is a right H^* -Galois and separable extension of C^H , a commutator right H^* -Galois extension if $V_B(B^H)$ is a right H^* -Galois extension of $(V_B(B^H))^H$ ([14]), a right H^* -Azumaya Galois extension if B is separable over B^H which is an Azumaya algebra over C^H ([9]), and a right H^* -DeMeyer-Kanzaki Galois extension if B is a right H^* -Azumaya Galois extension and $V_B(B^H) = C$ ([12]). Throughout, an H^* -Galois extension means a right H^* -Galois extension unless it is stated

otherwise.

Let P be a finitely generated and projective module over a commutative ring R . Then the rank function $\text{rank}_R(P)$ from prime $\text{Spec}(R)$ to the set of nonnegative integers is defined by $\text{rank}_R(P)(p) = \text{rank}_{R_p}(P_p)$ for each prime ideal $p \in \text{Spec}(R)$ where R_p is the local ring of R at p and $P_p (= P \otimes_R R_p)$ is a free module over R_p ([4], Corollary 4.11, page 31).

3. Characterizations

In this section, keeping all notations as given in section 2, we shall give two characterizations of a center H^* -Galois extension in terms of the smash product $B\#H$ and $V_B(B^H)\#H$ respectively. We begin with two lemmas.

Lemma 3.1. *If A_1 and A_2 are H^* -Galois extensions such that $A_1^H = A_2^H$ and $A_1 \subset A_2$, then $A_1 = A_2$.*

Proof. See Lemma 3.1 in [13].

Lemma 3.2. *If $V_B(B^H)$ is an H^* -Galois extension of $(V_B(B^H))^H$, then $(V_B(B^H))^H = C^H$ and $\text{rank}_{C^H}(V_B(B^H)) = \dim_k(H)$.*

Proof. Since $V_B(B^H)$ is an H^* -Galois extension of $(V_B(B^H))^H$ and both B and $B^H \cdot V_B(B^H)$ contain $V_B(B^H)$, both B and $B^H \cdot V_B(B^H)$ contain the H^* -Galois system of $V_B(B^H)$. Hence B and $B^H \cdot V_B(B^H)$ are also H^* -Galois extension of B^H . Since $B^H \cdot V_B(B^H) \subset B$, $B^H \cdot V_B(B^H) = B$ by Lemma 3.1. This implies that the center of B^H is C^H . But $(V_B(B^H))^H = V_{B^H}(B^H)$ which is the center of B^H . Thus $(V_B(B^H))^H = C^H$. Moreover, since $V_B(B^H)$ is an H^* -Galois extension of C^H ,

$$V_B(B^H)\#H \cong \text{Hom}_{C^H}(V_B(B^H), V_B(B^H))$$

where $V_B(B^H)$ is a finitely generated and projective C^H -module ([9], Theorem 2.2). Thus the rank of $V_B(B^H)$ over C^H is defined and $\text{rank}_{C^H}(V_B(B^H)) = \dim_k(H)$.

Theorem 3.3. *Let B be a left H -module algebra with center C . Then B is a center H^* -Galois extension of B^H if and only if $V_B(B^H)\#H$ is an Azumaya C^H -algebra and contains C as a maximal commutative separable subalgebra.*

Proof. (\implies) Since C is an H^* -Galois and separable extension of C^H such that $C \subset B^H C \subset B$, both $B^H C$ and B are H^* -Galois extensions of B^H and $B^H C = B$ by Lemma 3.1. Hence $V_B(B^H) = C$; and so $V_B(B^H)\#H = C\#H \cong \text{Hom}_{C^H}(C, C)$ where C is a finitely generated and projective C^H -module ([9], Theorem 2.2). But C^H is commutative with 1, so C is a progenerator of C^H . Thus $\text{Hom}_{C^H}(C, C)$ is an Azumaya C^G -algebra ([4], Proposition 4.1, page 56); and so $V_B(B^H)\#H$ is an Azumaya C^H -algebra. Moreover, since $C \subset \text{Hom}_{C^H}(C, C)$ by the left multiplication map, we can check that $V_{\text{Hom}_{C^H}(C, C)}(C) = C$. Therefore C is a maximal commutative separable subalgebra of $\text{Hom}_{C^H}(C, C)$; and so $V_B(B^H)\#H$ is an Azumaya C^H -algebra and contains C as a maximal commutative separable subalgebra.

(\impliedby) Since $V_B(B^H)\#H$ is an Azumaya C^H -algebra and finitely generated and projective over $V_B(B^H)$, $V_B(B^H)\#H$ is a Hirata separable extension of $V_B(B^H)$ ([7], Theorem 1). Noting that $V_B(B^H)$ is a progenerator of $V_B(B^H)$, we have that $V_B(B^H)$ is a progenerator of $V_B(B^H)\#H$ ([7], Lemma). Hence $V_B(B^H)$ is an H^* -Galois extension of $(V_B(B^H))^H$ ([9], Theorem 2.2). Thus $V_B(B^H)$ is an H^* -Galois extension of C^H and $\text{rank}_{C^H}(V_B(B^H)) = \dim_k(H)$ by Lemma 3.2. Moreover, C is a maximal commutative separable subalgebra of the Azumaya C^H -algebra $V_B(B^H)\#H$, so

$$C \otimes_{C^H} (V_B(B^H)\#H) \cong \text{Hom}_C(V_B(B^H)\#H, V_B(B^H)\#H)$$

(see the proof of Theorem 5.5 on page 64 in [4]). On the other hand, $V_B(B^H)$ is an H^* -Galois extension of C^H , so $V_B(B^H)\#H \cong \text{Hom}_{C^H}(V_B(B^H), V_B(B^H))$. Thus

$$C \otimes_{C^H} (V_B(B^H)\#H) \cong \text{Hom}_C(C \otimes_{C^H} V_B(B^H), C \otimes_{C^H} V_B(B^H)).$$

But C is a separable subalgebra of the Azumaya C^H -algebra $V_B(B^H)\#H$, so C is a direct summand of $V_B(B^H)\#H$ as a C -bimodule. This implies that C is a finitely generated and

projective C^H -module; and so $\text{rank}_{C^H}(C)$ is defined. Therefore, from the isomorphism

$$\begin{aligned} \text{Hom}_C(V_B(B^H)\#H, V_B(B^H)\#H) &\cong C \otimes_{C^H} (V_B(B^H)\#H) \\ &\cong \text{Hom}_C(C \otimes_{C^H} V_B(B^H), C \otimes_{C^H} V_B(B^H)) \end{aligned}$$

and $\text{rank}_{C^H}(V_B(B^H)) = \dim_k(H)$, we have that $\text{rank}_{C^H}(C) = \dim_k(H)$. Again, since C is a direct summand of $V_B(B^H)\#H$ as a C -bimodule, C is a direct summand of $V_B(B^H)$ as a C -bimodule. Thus $V_B(B^H) = C \oplus P$ for some finitely generated and projective C^H -module P . Consequently, that $\text{rank}_{C^H}(V_B(B^H)) = \text{rank}_{C^H}(C)$ implies that $\text{rank}_{C^H}(P) = 0$. Thus $P = 0$, and so $V_B(B^H) = C$. This completes the proof.

By Theorem 3.3, a center H^* -Galois extension can be characterized in terms of the smash product $B\#H$. The following lemma is important.

Lemma 3.4. *If A is an Azumaya C -algebra, then $D \otimes_C A$ is a Hirata separable extension of D for any C -algebra D .*

Proof. See Proposition 3.4 in [6].

Theorem 3.5. *Let B be a left H -module algebra with center C . Then B is a center H^* -Galois extension of B^H if and only if $B\#H$ is a projective Hirata separable extension of B^H containing B^H as a direct summand as a B^H -bimodule and C is a maximal commutative separable subalgebra of $V_B(B^H)\#H$ over C^H .*

Proof. (\implies) Since C is an H^* -Galois and separable extension of C^H , $B \cong B^H \otimes_{C^H} C$ such that $V_B(B^H) = C$ by Theorem 3.3. Hence $B\#H \cong (B^H \otimes_{C^H} C)\#H \cong B^H \otimes_{C^H} (C\#H)$ such that $C\#H$ is an Azumaya C^H -algebra by Theorem 3.3 again. Thus $B\#H$ is a projective Hirata separable extension of B^H by Lemma 3.4. Moreover, since C^H is a direct summand of $C\#H$ as a C^H -bimodule ([4], Lemma 3.1, page 51), B^H is a direct summand of $B^H \otimes_{C^H} (C\#H)$ as a B^H -bimodule. Thus B^H is a direct summand of $B\#H$ as a B^H -bimodule. Furthermore, C is an H^* -Galois and separable extension of C^H , so C is a maximal commutative separable subalgebra of $C\#H$ by Theorem 3.3.

(\Leftarrow) Since $B\#H$ is a projective Hirata separable extension of B^H and contains B^H as a direct summand as a B^H -bimodule, $V_{B\#H}(B^H)$ is a separable Z -algebra such that $V_{B\#H}(V_{B\#H}(B^H)) = B^H$ where Z is the center of $B\#H$ ([10], Proposition 1.3 and Proposition 1.4). This implies that the center Z of $B\#H$ is C^H . Thus $V_B(B^H)\#H$ ($= V_{B\#H}(B^H)$) is an Azumaya C^H -algebra. Moreover, by hypothesis, C is a maximal commutative separable subalgebra of $V_B(B^H)\#H$, so B is a center H^* -Galois extension of B^H by Theorem 3.3.

4. Properties

In this section, we shall give several properties of the smash products $B\#H$ and $V_B(B^H)\#H$ respectively where B is a center H^* -Galois extension of B^H . As given in section 3 in [15], let C be a commutative ring with 1 and A an R -algebra containing C which is an H^* -Galois extension of C^H . Then A is a left H^* -module algebra under the Ulbrich-Miyashita action: $ha = \alpha^{-1}(h)a$ for each $h \in H^*$ and $a \in A$ where $\alpha : C \otimes_{C^H} C \rightarrow C\#H^*$ by $\alpha(x \otimes y) = x\rho(y)$ with the structure map $\rho : C \rightarrow C\#H^*$ such that $C \subset A^{H^*}$, and it was shown that A is an H -Galois extension of C if and only if A is an Azumaya R -algebra, C a maximal commutative separable subalgebra of A , and A a finitely generated and projective C -module ([15], Theorem 3.1). Now by Theorem 3.3, let B be a center H^* -Galois extension of B^H . Then $V_B(B^H)\#H$ ($= C\#H$) is an Azumaya C^H -algebra with a maximal commutative separable subalgebra C . Hence $V_B(B^H)\#H$ is a left H^* -module algebra under the Ulbrich-Miyashita action. By noting that $C\#H$ is a left finitely generated and projective C -module, we have an H -Galois structure on $C\#H$. We note that any H^* -Galois extension B of B^H is also a Frobenius extension of B^H ([9], Corollary 5.2).

Theorem 4.1. *If B is a center H^* -Galois extension of B^H , then $V_B(B^H)\#H$ ($= C\#H$) is an H -Galois extension of C under the Ulbrich-Miyashita left action of H^* .*

Corollary 4.2. *If B is a center H^* -Galois extension of B^H , then $B\#H$ is an H -Galois extension of B under the Ulbrich-Miyashita left action of H^* .*

Proof. By Theorem 3.5, $B\#H \cong (B^H \otimes_{C^H} V_B(B^H))\#H \cong B^H C\#H$, so

$(B\#H)^{H^*} = B^H C = B$. Since $V_B(B^H)\#H$ is an H -Galois extension of C , $B\#H$ is an H -Galois extension of B .

Theorem 4.3. *If B is a center H^* -Galois extension of B^H , then*

(1) $B\#H$ is a Hirata separable extension of B ,

(2) $V_{B\#H}(B) = C$, and

(3) $V_{B\#H}(C) = B$.

Proof. (1) Since B is a center H^* -Galois extension of B^H , $B\#H \cong B^H \otimes_{C^H} (C\#H)$ by Theorem 3.5 such that $C\#H$ is an Azumaya C^H -algebra by Theorem 3.3. Noting that $C\#H$ is a left finitely generated and projective C -module, we have that $C\#H$ is a Hirata separable extension of C ([4], Theorem 1). But then $B\#H (\cong B^H \otimes_{C^H} (C\#H))$ is a Hirata separable extension of $B (\cong B^H \otimes_{C^H} C)$.

(2) Since $B = B^H C$, $V_{B\#H}(B) = V_{V_{B\#H}(B^H)}(C) = V_{C\#H}(C) = C$ by Theorem 3.3.

(3) Since B is a center H^* -Galois extension of B^H , $B\#H$ is a projective Hirata separable extension of B^H by Theorem 3.5. Moreover, since C is a separable extension of C^H , $B (\cong B^H \otimes_{C^H} C)$ is a separable extension of B^H . But B is a Frobenius extension of B^H ([9], Corollary 5.2), so B is a direct summand of $B\#H$ as a B -bimodule ([3], Theorem 1.8). Thus $V_{B\#H}(V_{B\#H}(B)) = B$ ([10], Proposition 1.4). Therefore, by part (2), $V_{B\#H}(B) = C$, we conclude that $V_{B\#H}(C) = B$.

Clearly, any H^* -DeMeyer-Kanzaki Galois extension is a center H^* -Galois extension, and any center H^* -Galois extension is a commutator H^* -Galois extension. Next we show a sufficient condition under which the converse holds for each inclusion.

Theorem 4.4. *Let B be a center H^* -Galois extension of B^H . Then B is an H^* -DeMeyer-Kanzaki Galois extension of B^H if and only if $B\#H$ is an Azumaya C^H -algebra.*

Proof. (\Leftarrow) Since C is an H^* -Galois and separable extension of C^H , it suffices to show that B^H is an Azumaya C^H -algebra. In fact, $B\#H \cong B^H \otimes_{C^H} (C\#H)$ such that $C\#H$ is an Azumaya C^H -algebra by Theorem 3.5 and Theorem 3.3. But $B\#H$ is an Azumaya C^H -algebra by hypothesis, so B^H is an Azumaya C^H -algebra by the commutator theorem for Azumaya algebras ([4], Theorem 4.3, page 57).

(\Rightarrow) Since B is an H^* -DeMeyer-Kanzaki Galois extension of B^H , B is an H^* -Azumaya Galois extension of B^H . Hence $B\#H$ is an Azumaya C^H -algebra ([9], Theorem 3.4).

Theorem 4.5. *Let B be a commutator H^* -Galois extension of B^H . Then B is a center H^* -Galois extension of B^H if and only if C is a maximal commutative separable subalgebra of $V_B(B^H)\#H$.*

Proof. (\Leftarrow) Since B is a commutator H^* -Galois extension of B^H , $V_B(B^H)$ is an H^* -Galois extension of $(V_B(B^H))^H$ such that $(V_B(B^H))^H = C^H$ by Lemma 3.2. Hence $V_B(B^H)\#H \cong \text{Hom}_{C^H}(V_B(B^H), V_B(B^H))$ such that $V_B(B^H)$ is a finitely generated and projective C^H -module. Noting that $V_B(B^H)$ is faithful over C^H , we have that $V_B(B^H)$ is a progenerator of C^H . Thus $\text{Hom}_{C^H}(V_B(B^H), V_B(B^H))$ is an Azumaya C^H -algebra; and so $V_B(B^H)\#H$ is an Azumaya C^H -algebra containing C as a maximal commutative separable subalgebra by hypothesis. Therefore B is a center H^* -Galois extension of B^H by Theorem 3.3.

(\Rightarrow) Since B is a center H^* -Galois extension of B^H , C is a maximal commutative separable subalgebra of $V_B(B^H)\#H$ by Theorem 3.3.

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