ON SEPARABLE SUBALGEBRAS OF A HOPF GALOIS AZUMAYA EXTENSION

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Abstract

Let B be an H^*-Galois Azumaya extension where H^* is the dual Hopf algebra of a finite dimensional Hopf algebra H over a field k. It is shown that any separable subalgebra A of B^H is the H-invariant subalgebra of a unique H^*-Galois Azumaya extension D in B such that D commutes with $V_{B^H}(A)$, the commutator subalgebra of A in B^H. Consequently, some correspondences between several classes of separable subalgebras of B are obtained.

1. Introduction

In [1] and [2], the class of Galois Azumaya extensions with Galois group G was studied, and a one-to-one correspondence was given between the set of separable subalgebras of B^G over C^G and the set of separable subalgebras of B containing $V_B(B^G)$ where C is the center of B and $V_B(B^G)$ is the commutator subring of B^G in B. In [4], the concept of a Galois Azumaya extension with Galois group G was generalized to an H^*-Galois Azumaya

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extension where H^* is the dual Hopf algebra of a finite dimensional Hopf algebra H over a field k, and a one-to-one correspondence was given between the set of separable subalgebras of B^H over C^H and the set of H^*-Galois Azumaya extensions in B containing $V_B(B^H)$ and separable over C^H. The purpose of the present paper is to give an expression for an H^*-Galois Azumaya extension B, $B^H \otimes_{C^H} V_B(B^H)$ where B^H is an Azumaya C^H-algebra and $V_B(B^H)$ is an H^*-Galois algebra over the center of B^H. We call B^H the base factor and $V_B(B^H)$ the Galois factor of B. Then we shall show that any separable subalgebra A of B^H over C^H is the H-invariant subalgebra of a unique H^*-Galois Azumaya extension D in B such that D commutes with $V_{B^H}(A)$. Consequently, we derive a one-to-one correspondence $\alpha : A \longrightarrow D$ from the set of separable subalgebras of B^H over C^H to the set of H^*-Galois Azumaya extensions in B which commutes with the commutator of its H-invariant subalgebra in B^H and separable over C^H. Moreover, the restriction of α to the set of Azumaya subalgebras of B^H over C^H induces a one-to-one correspondence between the set of Azumaya subalgebras of B^H over C^H and the set of H^*-Galois Azumaya extensions in B with the same Galois factor $V_B(B^H)$, and the restriction of α to the set of commutative separable subalgebras of B^H over C^H derives another one-to-one correspondence between the set of commutative separable subalgebras of B^H over C^H and the set of H^*-Galois algebras over Z in B of the form $V_B(V_{B^H}(Z))$ where Z is a commutative separable subalgebra of B^H.

2. Basic Definitions and Notations

Throughout, H denotes a finite dimensional Hopf algebra over a field k with comultiplication Δ and counit ε, H^* the dual Hopf algebra of H, B a left H-module algebra, C the center of B, $B^H = \{b \in B \mid hb = \varepsilon(h)b$ for all $h \in H\}$ which is called the H-invariant of B, and $B\#H$ the smash product of B with H where $B\#H = B \otimes_k H$ such that for all $b\#h$ and $b'\#h'$ in $B\#H$, $(b\#h)(b'\#h') = \sum b(h_1 b')\#h_2 h'$ where $\Delta(h) = \sum h_1 \otimes h_2$. B is called an H^*-Galois extension of B^H if B is a right H^*-comodule algebra with structure map $\rho : B \longrightarrow B \otimes_k H^*$ such that $\beta : B \otimes_{B^H} B \longrightarrow B \otimes_k H^*$ is a bijection where $\beta(a \otimes b) = (a \otimes 1)\rho(b)$.

For a subring A of B with the same identity 1, we denote the commutator subring of A in B by $V_B(A)$. we call B a separable extension of A if there exist $\{a_i, b_i$ in B, $i = 1, 2, ..., m$ for some integer $m\}$ such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where $\otimes$ is over A. An Azumaya algebra is a separable extension of its center. A ring B is called a Hirata separable extension of A if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B-bimodule. An H^*-Galois extension B is called an H^*-Galois Azumaya extension if B is separable over B^H which is an Azumaya algebra over C^H. An H^*-Galois Azumaya extension B is called an H^*-Galois algebra if $B^H = C^H$. We shall show that if B is an H^*-Galois Azumaya extension of B^H, then $B = B^H \cdot V_B(B^H) \cong B^H \otimes_{C^H} V_B(B^H)$ such that B^H is an Azumaya C^H-algebra and $V_B(B^H)$ is an H^*-Galois algebra over C^H. Then we call the H-invariant subalgebra B^H the base factor of B and the commutator of B^H in B, $V_B(B^H)$, the Galois factor of B for an H^*-Galois Azumaya extension B.

3. H^*-Galois Azumaya extensions

In this section, we shall show that any Azumaya subalgebra A of B^H over C^H is the H-invariant subalgebra of a unique H^*-Galois Azumaya extension in B with the same Galois factor $V_B(B^H)$, and give a one-to-one correspondence between the set of Azumaya subalgebras of B^H over C^H to the set of H^*-Galois Azumaya extensions in B with the same Galois factor $V_B(B^H)$. We begin with three lemmas.

Lemma 3.1. *If A_1 and A_2 are H^*-Galois Azumaya extensions such that $A_1^H = A_2^H$ and $A_1 \subset A_2$, then $A_1 = A_2$.*

Proof. By Theorem 5.1 in [4], there exist $\{x_i, y_i \in A_1 \,|\, i = 1, 2, ..., n\}$ for some integer n such that, for all $h \in H$, $\sum x_i(hy_i) = T(h)1_{A_1}$ where $T \in \int_{H^*}^r$, the set of right integrals in H^*. Let $t \in \int_H^l$, the set of left integrals in H, such that $T(t) = 1$, then $\{x_i, f_i = t(y_i -) \,|\, i = 1, 2, ..., n\}$ is a dual basis of the finitely generated and projective right module over A_1^H. Since $A_1 \subset A_2$ such that $A_1^H = A_2^H$, $\{x_i, f_i \,|\, i = 1, 2, ..., n\}$ is also a dual basis of the finitely generated and projective right module A_2 over A_1^H. This implies that $A_1 = A_2$.

Lemma 3.2. *If B is an H^*-Galois Azumaya extension of B^H, then $B = B^H \cdot V_B(B^H) \cong B^H \otimes_{C^H} V_B(B^H)$ such that B^H is an Azumaya C^H-algebra and $V_B(B^H)$ is an H^*-Galois algebra over C^H.*

Proof. Since B is an H^*-Azumaya Galois extension of B^H, $V_B(B^H)$ is an H^*-Azumaya Galois extension of $(V_B(B^H))^H$ ([4], Lemma 4.1) and B^H is an Azumaya C^H-algebra ([4], Theorem 3.4). Moreover, by the proof of Lemma 4.1 in [4], $B\#H$ is an Azumaya C^H-algebra such that $B\#H \cong B^H \otimes_{C^H} (V_B(B^H)\#H) \cong B^H \cdot (V_B(B^H)\#H)$ where B^H and $V_B(B^H)\#H$ are Azumaya C^H-algebras. But H is a finite dimensional Hopf algebra over a field k, so $B \cong B^H \otimes_{C^H} V_B(B^H)$ from the isomorphism $B\#H \cong B^H \otimes_{C^H} (V_B(B^H)\#H)$; and so $B = B^H \cdot V_B(B^H)$. Hence the center of $V_B(B^H)$ is C and $(V_B(B^H))^H = B^H \cap V_B(B^H) = C^H$. Thus $V_B(B^H)$ is an H^*-Galois algebra over C^H ([4], Lemma 4.1).

Let B be an H^*-Galois Azumaya extension of B^H and A an Azumaya subalgebra of B^H over C^H. Next we show that the Galois factor of the H^*-Galois Azumaya extension $V_B(A)$ is the same as the Galois factor of B.

Lemma 3.3. *Let B be an H^*-Galois Azumaya extension of B^H. If A is an Azumaya subalgebra of B^H over C^H, then $V_B(A)$ is an H^*-Galois Azumaya extension such that the base factor of $V_B(A)$ is $V_{B^H}(A)$ and the Galois factor of $V_B(A)$ is $V_B(B^H)$, that is, $(V_B(A))^H = V_{B^H}(A)$ and $V_{V_B(A)}((V_B(A))^H) = V_B(B^H)$, the same Galois factor of B.*

Proof. Since A is a separable subalgebra of B^H over C^H, by Lemma 4.1 in [4], $V_B(A)$ is an H^*-Galois Azumaya extension of $(V_B(A))^H$ which is the base factor of $V_B(A)$. Since $V_{B^H}(A) = B^H \cap V_B(A) = (V_B(A))^H$, the base factor of $V_B(A)$ is $V_{B^H}(A)$. Moreover, the Galois factor of $V_B(A)$ is $V_{V_B(A)}((V_B(A))^H) = V_{V_B(A)}(V_{B^H}(A)) \supset V_{V_B(A)}(B^H) = V_B(B^H) \cap V_B(A) = V_B(B^H)$. By Lemma 3.2, $V_B(A) \cong V_{B^H}(A) \otimes_Z V_{V_B(A)}((V_B(A))^H)$ where Z is the center of $V_{B^H}(A)$ and $V_{V_B(A)}((V_B(A))^H)$ is an H^*-Galois algebra over Z. But, by hypothesis, A is an Azumaya C^H-subalgebra of the Azumaya algebra B^H, so $V_{B^H}(A)$ is also an Azumaya C^H-subalgebra of B^H, that is, the center of $V_{B^H}(A)$ is C^H. Hence $Z = C^H$, and so $V_{V_B(A)}((V_B(A))^H)$ is an H^*-Galois algebra over C^H. Noting that

$V_B(B^H)$ is also an H^*-Galois algebra over C^H such that $V_B(B^H) \subset V_{V_B(A)}((V_B(A))^H)$, we conclude that $V_{V_B(A)}((V_B(A))^H) = V_B(B^H)$ by Lemma 3.1.

Next is the main theorem of this section.

Theorem 3.4. *Let B be an H^*-Galois Azumaya extension of B^H and A an Azumaya subalgebra of B^H over C^H. Then there exists a unique H^*-Galois Azumaya extension D contained in B such that $A = D^H$ (that is, A is the H-invariant subalgebra of a unique H^*-Galois Azumaya extension contained in B) and D commutes with $V_{B^H}(A)$. Moreover, D has Galois factor $V_B(B^H)$ and is separable over C^H.*

Proof. Let $A' = V_{B^H}(A)$. Then A' is an Azumaya subalgebra of B^H over C^H because A is so by hypothesis ([3], Theorem 4.3 on page 57). Thus $V_B(A')$ is an H^*-Galois Azumaya extension with base factor $V_{B^H}(A')$ and Galois factor $V_B(B^H)$ by Lemma 3.3, that is, $V_B(A') \cong V_{B^H}(A') \otimes_{C^H} V_B(B^H)$. But $V_{B^H}(A') = V_{B^H}(V_{B^H}(A)) = A$ ([3], Theorem 4.3 on page 57). so $A = D^H$ where $D = V_B(A')$ which is an H^*-Galois Azumaya extension contained in B with Galois factor $V_B(B^H)$ and separable over C^H.

Next we show that D is a unique H^*-Galois Azumaya extension such that $A = D^H$ and D commutes with $V_{B^H}(A)$. Let E be an H^*-Galois Azumaya extension in B such that $A = E^H$ and E commutes with $V_{B^H}(A)$. We claim that $E = D$. In fact, by Lemma 3.2, $E = A \cdot V_E(A) \cong A \otimes_{C^H} V_E(A)$ such that $V_E(A)$ is an H^*-Galois algebra over C^H, so $V_{B^H}(A) \cdot V_E(A)$ is an H^*-Galois extension of $(V_{B^H}(A) \cdot V_E(A))^H$ such that $V_{B^H}(A) \cdot V_E(A) \subset V_B(A)$. Since A is a separable subalgebra of B^H, $V_B(A)$ is an H^*-Galois Azumaya extension of $(V_B(A))^H$. But $(V_B(A))^H = V_{B^H}(A) \subset (V_{B^H}(A) \cdot V_E(A))^H \subset (V_B(A))^H$, so $(V_B(A))^H = (V_{B^H}(A) \cdot V_E(A))^H$. Thus $V_{B^H}(A) \cdot V_E(A) = V_B(A)$ by Lemma 3.1. This implies that $V_E(A) \subset V_B(A) = V_{B^H}(A) \cdot V_B(B^H)$ where $V_{B^H}(A)$ is the base factor of $V_B(A)$ and $V_B(B^H)$ is the Galois factor of $V_B(A)$ by Lemma 3.3. But, by hypothesis, E commutes with $V_{B^H}(A)$, so $V_E(A) \subset V_B(B^H)$. Thus both $V_E(A)$ and $V_B(B^H)$ are H^*-Galois algebras over C^H such that $V_E(A) \subset V_B(B^H)$. Therefore $V_E(A) = V_B(B^H)$ by Lemma 3.1. Consequently, $E = A \cdot V_E(A) = A \cdot V_B(B^H) = D$.

Theorem 3.4 gives that the map $\alpha : A \longrightarrow D \ (= V_B(V_{B^H}(A)))$ is a one-to-one correspondence between the set of Azumaya subalgebras of B^H over C^H and the set of H^*-Galois Azumaya extensions in B with Galois factor $V_B(B^H)$.

Corollary 3.5. *The map $\alpha : A \longrightarrow D$ as given in Theorem 3.4 is a one-to-one correspondence between the set of Azumaya subalgebras of B^H over C^H and the set of H^*-Galois Azumaya extensions in B with Galois factor $V_B(B^H)$.*

Proof. By Theorem 3.4, $A = D^H$, so α is a one-to-one. Moreover, for any H^*-Galois Azumaya extension E with Galois factor $V_B(B^H)$, $E \cong E^H \otimes_Z V_E(E^H)$ where Z is the center of E^H by Lemma 3.2 and $V_E(E^H) = V_B(B^H)$. Hence the center of $V_E(E^H)$ is C from the expression of B as given in Lemma 3.2. Thus the center of E is C. Therefore $Z = C^H$; and so E^H is an Azumaya C^H-subalgebra of B^H such that $\alpha : E^H \longrightarrow E$, that is, α is onto.

4. More correspondences

In this section, we shall show a more general correspondence than the one as given in Corollary 3.5. This gives a further description of the correspondence as given in [4]. In particular, a correspondence is derived for the set of commutative separable subalgebras of B^H over C^H. Let A be a separable subalgebra of B^H over C^H with center Z. Then it is known that A is an Azumaya Z-algebra in B^H and Z is a separable subalgebra of B^H over C^H. By the argument as given in Theorem 3.4, we can show that there exists a unique H^*-Galois Azumaya extension D contained in B such that $A = D^H$ and D commutes with $V_{B^H}(A)$.

Lemma 4.1. *Let B be an H^*-Galois Azumaya extension of B^H and A a separable subalgebra of B^H over C^H with center Z. Then $V_B(A)$ is an H^*-Galois Azumaya extension such that the base factor of $V_B(A)$ is $V_{B^H}(A)$ and the Galois factor of $V_B(A)$ is $V_B(V_{B^H}(Z))$, that is, $(V_B(A))^H = V_{B^H}(A)$ and $V_{V_B(A)}((V_B(A))^H) = V_B(V_{B^H}(Z))$.*

Proof. Since A is a separable subalgebra of B^H over C^H, $V_B(A)$ is an H^*-Galois Azumaya extension of $(V_B(A))^H$ containing $V_B(B^H)$ and separable over C^H ([4], Lemma 4.1). Let $F = V_B(A)$. Then $F = F^H \cdot V_F(F^H) \cong F^H \otimes V_F(F^H)$ by Lemma 3.2 where $\otimes$ is over the center of F^H. Since $F^H = (V_B(A))^H = B^H \cap V_B(A) = V_{B^H}(A)$, the base factor of F (($= V_B(A)$)) is $V_{B^H}(A)$. Noting that A is a separable subalgebra of the Azumaya algebra B^H over C^H, we have that A and $V_{B^H}(A)$ ($= F^H$) have the same center Z. Hence the Galois factor of F, $V_F(F^H)$, is an H^*-Galois algebra over Z by Lemma 3.2. Moreover, $V_F(F^H) = V_{V_B(A)}((V_B(A))^H) = V_{V_B(A)}(V_{B^H}(A)) \supset V_{V_B(A)}(V_{B^H}(Z)) = V_B(V_{B^H}(Z)) \cap V_B(A) = V_B(V_{B^H}(Z))$ because A is an Azumaya Z-algebra contained in B^H. Thus $V_B(V_{B^H}(Z)) \subset V_F(F^H)$. Noting that Z is a separable subalgebra of the Azumaya algebra B^H over C^H, we have that $V_{B^H}(Z)$ is a separable subalgebra of B^H over C^H ([3], Theorem 4.3 on page 57). Hence $V_B(V_{B^H}(Z))$ is an H^*-Galois Azumaya extension of $(V_B(V_{B^H}(Z)))^H$ ([4], Lemma 4.1). But $(V_B(V_{B^H}(Z)))^H = V_{B^H}(V_{B^H}(Z)) = Z$, so Both $V_F(F^H)$ and $V_B(V_{B^H}(Z))$ are H^*-Galois algebra over Z such that $V_B(V_{B^H}(Z)) \subset V_F(F^H)$. Thus, by Lemma 3.1, $V_B(V_{B^H}(Z)) = V_F(F^H)$, the Galois factor of F ($= V_B(A)$).

Next is a generalization of Theorem 3.4.

Theorem 4.2. *Let B be an H^*-Galois Azumaya extension of B^H and A a separable subalgebra of B^H over C^H with center Z. Then there exists a unique H^*-Galois Azumaya extension D such that $A = D^H$ and D commutes with $V_{B^H}(A)$.*

Proof. Let $A' = V_{B^H}(A)$. Then A' is a separable subalgebra of B^H over C^H with the same center Z of A ([3], Theorem 4.3 on page 57). Thus $V_B(A')$ is an H^*-Galois Azumaya extension with base factor $V_{B^H}(A')$ and Galois factor $V_B(V_{B^H}(Z))$ by Lemma 4.1, that is, $V_B(A') \cong V_{B^H}(A') \otimes_Z V_B(V_{B^H}(Z))$. But $V_{B^H}(A') = V_{B^H}(V_{B^H}(A)) = A$. so $A = D^H$ where $D = V_B(A') = V_B(V_{B^H}(A))$ which is an H^*-Galois Azumaya extension with Galois factor $V_B(V_{B^H}(Z))$.

Next we show that D is a unique H^*-Galois Azumaya extension such that $A = D^H$ and D commutes with $V_{B^H}(A)$. Let E be an H^*-Galois Azumaya extension such that

$A = E^H$ and E commutes with $V_{B^H}(A)$. Then $E = A \cdot V_E(A) \cong A \otimes_Z V_E(A)$ such that $V_E(A)$ is an H^*-Galois algebra over Z by Lemma 3.2. Hence $V_{B^H}(A) \cdot V_E(A)$ is an H^*-Galois extension of $(V_{B^H}(A) \cdot V_E(A))^H$ such that $V_{B^H}(A) \cdot V_E(A) \subset V_B(A)$. Since A is a separable subalgebra of B^H, $V_B(A)$ is an H^*-Galois Azumaya extension of $(V_B(A))^H$. But $(V_B(A))^H = V_{B^H}(A) \subset (V_{B^H}(A) \cdot V_E(A))^H \subset (V_B(A))^H$, so $(V_B(A))^H = (V_{B^H}(A) \cdot V_E(A))^H$. Thus $V_{B^H}(A) \cdot V_E(A) = V_B(A)$ by Lemma 3.1; and so $V_E(A) \subset V_B(A)$. Moreover, by Lemma 4.1, the H^*-Galois Azumaya extension $V_B(A)$ has base factor $V_{B^H}(A)$ and Galois factor $V_B(V_{B^H}(Z))$. But then, by hypothesis, E commutes with $V_{B^H}(A)$, so $V_E(A) \subset V_B(V_{B^H}(Z))$. Thus both $V_E(A)$ and $V_B(V_{B^H}(Z))$ are H^*-Galois algebras over Z such that $V_E(A) \subset V_B(V_{B^H}(Z))$. Therefore $V_E(A) = V_B(V_{B^H}(Z))$ by Lemma 3.1. Consequently, $E = A \cdot V_E(A) = A \cdot V_B(V_{B^H}(Z)) = D$.

By the similar argument as given in Corollary 3.5, we have the following one-to-one correspondence $\alpha : A \longrightarrow D$ from the set of separable subalgebras of B^H over C^H to the set of H^*-Galois Azumaya extensions in B which commutes with the commutator of its H-invariant subalgebra in B^H and separable over C^H.

Corollary 4.3. *The map $\alpha : A \longrightarrow D$ $(= V_B(V_{B^H}(A)))$ as given in Theorem 4.2 is a one-to-one correspondence between the set of separable subalgebras of B^H over C^H and the set of H^*-Galois Azumaya extensions in B which commutes with the commutator of its H-invariant subalgebra in B^H and separable over C^H.*

We remark that Lemma 4.1, Theorem 4.2, and Corollary 4.3 give a further description of $V_B(A)$ for a separable subalgebra A of B^H in the one-to-one correspondence theorem as given by Theorem 4.3 in [4] between the set of separable subalgebras of B^H over C^H and the set of H^*-Galois Azumaya extensions in B containing $V_B(B^H)$ and separable over C^H.

Theorem 4.2 also induces a one-to-one correspondence between the set of commutative separable subalgebras of B^H over C^H and the set of H^*-Galois algebras in B of the form $V_B(V_{B^H}(Z))$ where Z is a commutative separable subalgebra of B^H.

Corollary 4.4. *The restriction of α as given in Theorem 4.2 to the set of commutative separable subalgebras of B^H over C^H is a one-to-one correspondence between the set of commutative separable subalgebras of B^H over C^H and the set of H^*-Galois algebras in B of form $V_B(V_{B^H}(Z))$ over Z which is separable over C^H.*

Proof. For a commutative separable subalgebra A of B^H, by Theorem 4.2, $\alpha(A) = A \otimes_Z V_B(V_{B^H}(A)) \cong V_B(V_{B^H}(Z))$ (for $A = Z$) which is an H^*-Galois Z-algebras in B. Hence α is well defined and is one-to-one. Moreover, let D be an H^*-Galois algebra in B of form $V_B(V_{B^H}(Z))$ where Z is a commutative separable subalgebra of B^H. Then $D^H = (V_B(V_{B^H}(Z)))^H = V_{B^H}(V_{B^H}(Z)) = Z$ because Z is a separable subalgebra of the Azumaya algebra B^H. Thus $\alpha : Z \longrightarrow D = V_B(V_{B^H}(Z))$ is onto.

Remark. Let $S(Z) = \{A \,|\, A$ is a separable subalgebra of B^H with center $Z\}$. Then the set of all separable subalgebra of B^H is $\cup\{S(Z) \,|\, Z$ is a commutative separable subalgebra of $B^H\}$. Corollary 4.3 and Corollary 4.4 imply that the map $\beta : S(Z) \longrightarrow V_B(V_{B^H}(Z))$ is a one-to-one correspondence between the set $\{S(Z) \,|\, Z$ is a commutative separable subalgebra of $B^H\}$ and the set of H^*-Galois algebras of the form $V_B(V_{B^H}(Z))$ over Z which is separable over C^H.

REFERENCES

[1] R. Alfaro and G. Szeto, Skew Group Rings which are Azumaya, *Comm. in Algebra,* **23** (1995), no. 6, 2255-2261.

[2] R. Alfaro and G. Szeto, On Galois Extensions of an Azumaya Algebra, *Comm. in Algebra,* **25** (1997), no. 6, 1873-1882.

[3] F. R. DeMeyer and E. Ingraham, *Separable algebras over commutative rings,* Volume 181. Springer Verlag, Berlin, Heidelberg, New York, 1971.

[4] M. Ouyang, Azumaya Extensions and Galois Correspondence, *Algebra Colloquium,* **7** (2000), no. 1, 43-57.