The Galois and Automorphism Groups of a Semiconnected Galois Extension

George Szeto and **Lianyong Xue**

Department of Mathematics, Bradley University
Peoria, Illinois 61625 – U.S.A.
Email: szeto@bradley.edu and lxue@bradley.edu

Abstract

Let B be a Galois extension with Galois group G with finitely many central idempotents. Then the Galois groups which are isomorphic with G are computed, and a relation between $\mathrm{Aut}_{B^G}(B)$ and G is also obtained.

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1. Introduction. The Galois theory for Galois extensions of rings with no idempotents but 0 and 1 (connected rings) and with finitely many central idempotents (semiconnected rings) were studied ([2], [3], [4], [6]), and the general Galois extensions of rings were investigated ([1], [5], [7]). Let B be a connected commutative Galois extension with Galois group G. It was shown that $G = \mathrm{Aut}_{B^G}(B)$, the automorphism group of B ([2], Theorem 3.5). This extends the fact for Galois extensions for fields. In the present paper, we are interested in the Galois groups of a semiconnected Galois extension and its relation with the automorphism group of the extension. Let B be a semiconnected Galois extension of B^G with Galois group G, $\{e_i \,|\, i = 1, 2, \cdots, m$ for some integer $m\}$ the set of minimal central idempotents of B, and $\{O_j \,|\, j = 1, 2, \cdots, t$ for some integer $t\}$ the set of orbits of $\{e_i\}$ under the G-action. Then for any $e_i, e_k \in O_j$, there exists a $g \in G$

such that $g(e_i) = e_k$. Let $E_j = \sum_{e_i \in O_j} e_i$. We have $g(E_j) = E_j$, a minimal central idempotent in B^G for each $g \in G$, $E_j E_l = E_j \delta_{jl}$, and $\sum_{j=1}^{t} E_j = 1$. This implies that $B = \oplus \sum_{j=1}^{t} BE_j$. We shall compute all Galois groups G' of B which are isomorphic with G such that $B^{G'} = B^G$. Moreover, for a semiconnected commutative Galois extension B with Galois group G, an expression of the automorphism group $\mathrm{Aut}_{B^G}(B)$ is given in terms of $G(e_1)$ where $G(e_1) = \{g \in G \,|\, g(e_1) = e_1\}$. This generalizes the fact that $G = \mathrm{Aut}_{B^G}(B)$ for a connected commutative Galois extension.

2. Definitions and Notations. Let B be a ring with 1, G a finite automorphism group of B, and B^G the set of elements in B fixed under each element in G. Then B is called a Galois extension of B^G with Galois group G if there exist elements $\{a_i, b_i$ in B, $i = 1, 2, ..., k\}$ for some integer k such that $\sum_{i=1}^{k} a_i g(b_i) = \delta_{1,g}$ for each $g \in G$. Such a set $\{a_i, b_i\}$ is called a G-Galois system for B. We call B connected if it contains no central idempotents but 0 and 1, and semiconnected if it contains only finitely many central idempotents.

Throughout this paper, the ring B is a semiconnected Galois extension of B^G with Galois group G, C the center of B with finite number of idempotents, $\{e_i \,|\, i = 1, 2, \cdots, m$ for some integer $m\}$ the set of minimal central idempotents of B, $G(e_i) = \{g \in G \,|\, g(e_i) = e_i\}$ for each e_i, and $G(e_i, e_j) = \{g \in G \,|\, g(e_i) = e_j\}$ for $i \neq j$. The fat group of G is denoted by $\overline{G}$, that is, $\overline{G} = \{\alpha \in \mathrm{Aut}_{B^G}(B)$ such that for each $i = 1, 2, \cdots, m$, $\alpha|_{Be_i} = g_i|_{Be_i}$ for some $g_i \in G\}$. By a Galois extension B with isomorphic Galois groups G and G', we means that $G \cong G'$ such that $B^G = B^{G'}$.

3. Isomorphic Galois Groups. In this section, let B be a semiconnected Galois extension (not necessarily commutative) of B^G with Galois group G, $\{e_i \,|\, i = 1, 2, \cdots, m$ for some integer $m\}$ the set of minimal central idempotents of B, $\{O_j \,|\, j = 1, 2, \cdots, t$ for some integer $t\}$ the set of orbits of $\{e_i\}$ under the G-action, and $E_j = \sum_{e_i \in O_j} e_i$. Then $B = \oplus \sum_{j=1}^{t} BE_j$. We shall compute all Galois groups G' of B isomorphic with G, that is, B is a Galois extension of a subring A with Galois groups $G' \cong G$. We begin with some properties of BE_j.

Proposition 3.1. *By keeping the above notation, we have that*

(1) $\{E_j \,|\, j = 1, 2, \cdots, t\}$ *is the set of minimal central idempotents of* B^G*, and*

(2) if B *is a Galois extension of a subring* A *with Galois groups* $G' \cong G$*, then* G *and* G' *have the same orbits.*

Proof. (1) Assume E_j is not a minimal central idempotent in B^G. Then $E_j = F' + F''$ where F' and F'' are idempotents in B^G such that $F' = \sum_{i \in I} e_i$ and $F'' = \sum_{k \in K} e_k$ for some partition $\{e_i \,|\, i \in I\}$ and $\{e_k \,|\, k \in K\}$ of O_j. Since there exists a $g \in G$ such that $g(e_i) = e_k$, $g(F') \neq \sum_{i \in I} e_i = F'$. This contradicts to the fact $F' \in B^G$. Thus $\{E_j \,|\, j = 1, 2, \cdots, t\}$ is the set of minimal central idempotents summing to 1 in B^G.

(2) Since B is a Galois extension of a subring A with Galois group $G' \cong G$, $B^G = B^{G'} = A$. Let $\{E_j \,|\, j = 1, 2, \cdots, t\}$ be the set of minimal central idempotents of B^G derived from the orbits of $\{e_i\}$ under the G-action and $\{E'_k \,|\, k = 1, 2, \cdots, s\}$ be the set of minimal central idempotents of $B^{G'}$ derived from the orbits of $\{e_i\}$ under the G'-action. Since $B^G = B^{G'} = A$, $\{E_j \,|\, j = 1, 2, \cdots, t\} = \{E'_k \,|\, k = 1, 2, \cdots, s\}$ by part (1). Thus G and G' have the same orbits of $\{e_i \,|\, i = 1, 2, \cdots, m\}$.

Lemma 3.2. *By keeping the above notations,* $B = \oplus \sum_{j=1}^{t} BE_j$ *such that* BE_j *is a Galois extension of* $B^G E_j$ *with Galois group* $G|_{BE_j}$ *which is isomorphic with* G.

Proof. Let $\{a_i, b_i$ in B, $i = 1, 2, ..., p$ for some integer $p\}$ be a Galois system for B. Then $\sum_{i=1}^{p} a_i g(b_i) = \delta_{1,g}$ for each $g \in G$. Since $g(E_j) = E_j$ for each $g \in G$, $\sum_{i=1}^{p} (a_i E_j) g(b_i E_j) = E_j \sum_{i=1}^{p} a_i g(b_i) = E_j \delta_{1,g}$ for each $g \in G$. If $g|_{BE_j} = 1$, then

$\sum_{i=1}^{p}(a_iE_j)g(b_iE_j) = E_j$. Hence $g = 1$ in G; for otherwise, $g \neq 1$, we have that $\sum_{i=1}^{p}(a_iE_j)g(b_iE_j) = E_j\sum_{i=1}^{p}a_ig(b_i) = 0$, a contradiction. This implies that $G \longrightarrow G|_{BE_j}$ by $g \longrightarrow g|_{BE_j}$ is an isomorphism for each j. Therefore, by noting that $\sum_{i=1}^{p}(a_iE_j)g(b_iE_j) = E_j\delta_{1,g}$ for each $g \in G$, we conclude that BE_j is a Galois extension with Galois group $G|_{BE_j}$ which is isomorphic with G. Moreover, since $g \longrightarrow g|_{BE_j}$ is an isomorphism from G to $G|_{BE_j}$ for each j, $(BE_j)^{G|_{BE_j}} \subset B^G$. Hence $(BE_j)^{G|_{BE_j}} = (BE_j)^{G|_{BE_j}}E_j \subset B^GE_j$. Clearly, $B^GE_j \subset (BE_j)^{G|_{BE_j}}$. Thus $B^GE_j = (BE_j)^{G|_{BE_j}}$. This completes the proof.

Now we derive an expression for each $g \in G$.

Theorem 3.3. *By keeping the notations of Lemma 3.2,* $G = \{g \in G \,|\, g = g_1 \oplus \sum_{j=2}^{t}\phi_j(g_1)$ *where* $g_1 = g|_{BE_1}$ *and* $\phi_j : G|_{BE_1} \longrightarrow G|_{BE_j}$ *is an isomorphism for each* $j = 2, 3, \cdots, t$.

Proof. Since $g(b) = g(\sum_{j=1}^{t}bE_j) = \sum_{j=1}^{t}g(bE_j)$ for each $b \in B$, $g = \oplus\sum_{j=1}^{t}g|_{BE_j}$. By Lemma 3.1, $\pi_j : G \longrightarrow G|_{BE_j}$ is an isomorphism. Let $\phi_j = \pi_j \cdot \pi_1^{-1}$. Then $\phi_j : G|_{BE_1} \longrightarrow G|_{BE_j}$ is an isomorphism such that $g(b) = \sum_{j=1}^{t}g(bE_j) = \sum_{j=1}^{t}\pi_j(g)(bE_j) = \sum_{j=1}^{t}(\phi_j\pi_1)(g)(bE_j) = \sum_{j=1}^{t}\phi_j(g_1)(bE_j)$ where $g_1 = g|_{BE_1} = \pi_1(g)$. Thus $g = g_1 \oplus \sum_{j=2}^{t}\phi_j(g_1)$.

Theorem 3.3 implies that a Galois group G induces $t-1$ isomorphism $\phi_j : G|_{BE_1} \longrightarrow G|_{BE_j}$ for $j = 2, 3, \cdots, t$. The converse also holds.

Theorem 3.4. *Let* $\phi'_j : G|_{BE_1} \longrightarrow G|_{BE_j}$ *be a group isomorphism for* $j = 2, 3, \cdots, t$. *Then* $G' = \{g_1 \oplus \sum_{j=2}^{t}\phi'_j(g_1) \,|\, g_1 \in G|_{BE_1}\}$ *is a Galois group for* B *which is isomorphism with* G*, that is,* $G \cong G'$ *such that* $B^G = B^{G'}$.

Proof. Since $\phi'_j : G|_{BE_1} \longrightarrow G|_{BE_j}$ is an isomorphism and $G|_{BE_1}$ is a group, it is straightforward to check that G' is a group under the componentwise operation. Moreover, $\alpha : G|_{BE_1} \longrightarrow G'$ by $\alpha(g_1) = g_1 \oplus \sum_{j=2}^{t} \phi'_j(g_1)$ for each $g_1 \in G|_{BE_1}$ is a group isomorphism. In fact, for $g_1, g'_1 \in G|_{BE_1}$,

$$\alpha(g_1g'_1) = g_1g'_1 \oplus \sum_{j=2}^{t} \phi'_j(g_1g'_1) = \big(g_1 \oplus \sum_{j=2}^{t} \phi'_j(g_1)\big)\big(g'_1 \oplus \sum_{j=2}^{t} \phi'_j(g'_1)\big) = \alpha(g_1)\alpha(g'_1).$$

Also $\alpha(g_1) = 1$ implies that $g_1 = 1$, so α is an isomorphism. Thus $G \cong G|_{BE_1} \cong G'$. Next, let $b \in B^{G'}$. Then for each $g_1 \in G|_{BE_1}$, $\oplus \sum_{j=1}^{t} bE_j = b = (g_1 \oplus \sum_{j=2}^{t} \phi'_j(g_1))(b) = g_1(bE_1) \oplus \sum_{j=2}^{t} \phi'_j(g_1)(bE_j)$. Hence $g_1(bE_1) = bE_1$ and $\phi'_j(g_1)(bE_j) = bE_j$ for $j = 2, 3, \cdots, t$. But $\phi'_j(G|_{BE_1}) = G|_{BE_j}$, so $bE_j \in (BE_j)^{G|_{BE_j}} = B^G E_j$ by Lemma 3.2 for each $j = 1, 2, \cdots, t$. Thus $b = \sum_{j=1}^{t} bE_j \in B^G$; and so $B^{G'} \subset B^G$. Also, by the similar argument, $B^G \subset B^{G'}$. Thus $B^G = B^{G'}$. Moreover, it is easy to check that a G-Galois system for B is a G'-Galois system for B, so G' is also a Galois group for B.

By Theorem 3.3, we have a map $\alpha : G' \longrightarrow (1, \phi'_2, ..., \phi'_t)$ from the set of Galois groups isomorphic with G such that $G|_{BE_1} = G'|_{BE_1}$ to the set of $1 \oplus \big(\oplus \sum_{j=2}^{t} \mathrm{Iso}(G|_{BE_1}, G|_{BE_j})\big)$ where $\phi'_j : G|_{BE_1} \longrightarrow G|_{BE_j}$ is an isomorphism. Thus Theorem 3.4 implies that α is a surjection. Next we want to show that α is a bijection.

Theorem 3.5. *$\alpha : G' \longrightarrow (1, \phi'_2, ..., \phi'_t)$ is a bijection from the set of Galois groups isomorphic with G such that $G|_{BE_1} = G'|_{BE_1}$ to the set of $1 \oplus \big(\oplus \sum_{j=2}^{t} \mathrm{Iso}(G|_{BE_1}, G|_{BE_j})\big)$ where $\phi'_j : G|_{BE_1} \longrightarrow G|_{BE_j}$ is an isomorphism.*

Proof. It suffices to show that α is an injection. Let G' and G'' be Galois groups isomorphic with G. Then $G' = \{g_1 \oplus \sum_{j=2}^{t} \phi'_j(g_1) \,|\, g_1 \in G|_{BE_1}\}$ and $G'' = \{g_1 \oplus \sum_{j=2}^{t} \phi''_j(g_1) \,|\, g_1 \in G|_{BE_1}\}$ for some $\phi'_j, \phi''_j \in \mathrm{Iso}(G|_{BE_1}, G|_{BE_j})$. If $(1, \phi'_2, ..., \phi'_t) = (1, \phi''_2, ..., \phi''_t)$, then $\phi'_j = \phi''_j$ for each $j = 2, 3, \cdots, t$. Thus $G' = G''$; and so α is an injection. Also, by Theore 3.4, α is a surjection, so α is a bijection.

As a consequence of Theorem 3.5, we obtain the number of Galois groups G' isomorphic with G such that $G|_{BE_1} = G'|_{BE_1}$.

Corollary 3.6. *Let B be a semiconnected Galois extension with Galois group G. Then there are $|Aut(G)|^{t-1}$ Galois groups G' for B isomorphic with G such that $G|_{BE_1} = G'|_{BE_1}$.*

Proof. By Lemma 3.2, $G|_{BE_j} \cong G$ for each $j = 1, 2, 3, \cdots, t$, so $|\mathrm{Iso}(G|_{BE_1}, G|_{BE_j})| = |\mathrm{Aut}(G)|$, that is, the order of $\mathrm{Aut}(G)$ is equal to the number of isomorphisms in $\mathrm{Iso}(G|_{BE_1}, G|_{BE_j})$. Thus the number of Galois groups for B isomorphic with G is equal to $|\mathrm{Aut}(G)|^{t-1}$ by Theorem 3.5.

Now we consider any Galois group $G' \cong G$ for B.

Corollary 3.7. *Let B be a semiconnected Galois extension with Galois group G. Then there are $|Aut(G)|^t$ Galois groups G' for B isomorphic with G.*

Proof. Let G' be a Galois group for B isomorphic with G. Then $B^{G'} = B^G$ with minimal central idempotents $\{E_j \,|\, j = 1, 2, \cdots, t \text{ for some integer } t\}$ by Lemma 3.1 such that $B = \oplus \sum_{j=1}^t BE_j$. Hence $G' \cong G'|_{BE_1} \cong G|_{BE_1} \cong G$ by Lemma 3.2. Thus there are $|\mathrm{Aut}(G)| \cdot |\mathrm{Aut}(G)|^{t-1} = |\mathrm{Aut}(G)|^t$ number of isomorphic Galois groups with G for B by Corollary 3.6.

4. Galois and Automorphism Groups. Let B be a semiconnected Galois extension with Galois group G. Then $B = \oplus \sum_{j=1}^t BE_j$ as given in Lemma 3.1. Observing that $E_j = \sum_{e_i \in O_j} e_i$ where $\{O_j \,|\, j = 1, 2, \cdots, t \text{ for some integer } t\}$ is the set of orbits of the minimal central idempotents $\{e_i \,|\, i = 1, 2, \cdots, m \text{ for some integer } m\}$ of B under the G-action, we have a Galois extension BE_j of $B^G E_j$ with a transitive Galois group $G|_{BE_j}$ which is isomorphic with G, that is, for any $e_i, e_k \in O_j$, there exists a $g \in G$ such that $g(e_i) = e_k$. In this section, we shall show a relation between $\mathrm{Aut}_{B^G e_i}(Be_i)$ and $\mathrm{Aut}_{B^G E_j}(BE_j)$ for each $e_i \in O_j$. This derives a generalization of a well known property of a commutative connected Galois algebra B with Galois group G that $G = \mathrm{Aut}_{B^G}(B)$. We shall employ some results of K. Kishimoto and T. Nagahara ([4], Theorem 13 and 15).

Lemma 4.1. *Let B be a semiconnected Galois extension with a transitive Galois group G and $\{e_i \,|\, i = 1, 2, \cdots, m$ for some integer $m\}$ the set of minimal central idempotents of B. Assume that $G(e_1) \neq \{1\}$. Then $B = \oplus \sum_{i=1}^{m} Be_i$ such that*

(i) $Be_i \cong Be_j$ for all i and j,

(ii) $G(e_j) = gG(e_i)g^{-1}$ for some $g \in G$ such that $g(e_i) = e_j$,

(iii) Be_i is a Galois extension with Galois group $G(e_i)$.

Proof. This is a consequence of Theorem 13 and Theorem 15 in [4].

Let $\mathrm{Iso}(Be_i, Be_j)$ be the set of isomorphisms α_{ij} such that $\alpha_{ij}(ae_i) = ae_j$ for each $a \in B^G$ and $G(e_i, e_j)$ be the set of $g \in G$ such that $g(e_i) = e_j$. Then it is easy to check the following statements.

Lemma 4.2. *By keeping the notations of Lemma 4.1, we have*

(i) $\mathrm{Iso}(Be_i, Be_j) = \alpha \mathrm{Aut}_{(Be_i)^{G(e_i)}}(Be_i)$ *for some $\alpha \in Aut_{B^G}(B)$ such that $\alpha(e_i) = e_j$;*

(ii) $\mathrm{Iso}(Be_i, Be_k) = \alpha_{ij}\alpha_i \mathrm{Aut}_{(Be_1)^{G(e_1)}}(Be_1)\alpha_i^{-1}$ *for a $\alpha_i \in \mathrm{Iso}(Be_1, Be_i)$ and $\alpha_{ij} \in \mathrm{Iso}(Be_i, Be_k)$;*

(iii) $G(e_i, e_j) = gG(e_i)$ for some $g \in G$ such that $g(e_i) = e_j$;

(iv) $G(e_i, e_k) = gg_iG(e_1)g_i^{-1}$ for a $g_i \in G(e_1, e_i)$ and $g \in G(e_i, e_k)$.

Next, we give an expression for the elements in B^G.

Lemma 4.3. *By keeping the notations of Lemma 4.2,*

$$B^G = \{\oplus \sum_{i=1}^{m} g_i(be_1)e_i \,|\, be_1 \in (Be_1)^{G(e_1)} \text{ and } g_i \in G \text{ such that } g_i(e_1) = e_i\}.$$

Proof. For any $b \in B^G$, $b = \oplus \sum_{i=1}^{m} be_i$ such that $\oplus \sum_{i=1}^{m} be_i = b = g(b) = \oplus \sum_{i=1}^{m} g(be_i)g(e_i)$ for all $g \in G$. Since g permutes $\{e_i\}$, $be_i = g(be_1)e_i$ where $g(e_1) = e_i$. But G is a transitive Galois group, so there exists a $g_i \in G$ such that $g_i(e_1) = e_i$ for each $i = 1, 2, \cdots, m$. Thus $be_i = g_i(be_1)e_i$ for each $i = 1, 2, \cdots, m$; and so $b = \oplus \sum_{i=1}^{m} be_i = \oplus \sum_{i=1}^{m} g_i(be_1)e_i$ such that $be_1 \in (Be_1)^{G(e_1)}$. Therefore $B^G \subset \{\oplus \sum_{i=1}^{m} g_i(be_1)e_i \,|\, be_1 \in (Be_1)^{G(e_1)}$ and $g_i \in G$ such that $g_i(e_1) = e_i\}$. Conversely, let $x = \oplus \sum_{i=1}^{m} g_i(be_1)e_i$ where $be_1 \in (Be_1)^{G(e_1)}$ and $g_i \in G$ such that $g_i(e_1) = e_i$. Then for any $g \in G$, $g(x) = \oplus \sum_{i=1}^{m} gg_i(be_1)gg_i(e_1)$. Since g permutes $\{e_1, e_2, \cdots, e_m\}$, $gg_i(e_1) = g(e_i) = e_{\pi(i)}$ where π is a permutation of $\{1, 2, \cdots, m\}$ induced by g, that is, $gg_i \in G(e_1, e_{\pi(i)})$. Hence, by Lemma 4.2(iii), $gg_i = g_{\pi(i)}h_i$ for some $h_i \in G(e_1)$. Thus

$$\begin{aligned} g(x) &= \oplus \sum_{i=1}^{m} gg_i(be_1)gg_i(e_1) = \oplus \sum_{i=1}^{m} g_{\pi(i)}h_i(be_1)g_{\pi(i)}h_i(e_1) \\ &= \oplus \sum_{i=1}^{m} g_{\pi(i)}(be_1)g_{\pi(i)}(e_1) = \oplus \sum_{i=1}^{m} g_{\pi(i)}(be_1)e_{\pi(i)} \\ &= \oplus \sum_{i=1}^{m} g_i(be_1)e_i = x. \end{aligned}$$

This implies that $\{\oplus \sum_{i=1}^{m} g_i(be_1)e_i \,|\, be_1 \in (Be_1)^{G(e_1)}$ and $g_i \in G$ such that $g_i(e_1) = e_i\} \subset B^G$. Thus the proof is complete.

Lemma 4.3 implies that $B^Ge_i = (Be_i)^{G(e_i)}$ for each i, so

$$\mathrm{Iso}(Be_i, Be_k) = \alpha_{ij}\alpha_i \mathrm{Aut}_{B^Ge_1}(Be_1)\alpha_i^{-1}$$

for a $\alpha_i \in \mathrm{Iso}(Be_1, Be_i)$ and $\alpha_{ij} \in \mathrm{Iso}(Be_i, Be_k)$.

Lemma 4.4. *An automorphism of* B, $f \in \mathrm{Aut}_{B^G}(B)$ *if and only if* $f = \oplus\sum_{i=1}^m f_i$ *where* $f_i \in \mathrm{Iso}(Be_i, Be_{\pi(i)})$ *for some permutation* π *of* $\{1, 2, \ldots, m\}$.

Proof. Since $\{e_i\}$ is the set of minimal central idempotents of B, f permutes the set $\{e_i\}$. Hence $f|_{Be_i} \in \mathrm{Iso}(Be_i, Be_{\pi(i)})$ for each i for some permutation π of $\{1, 2, \ldots, m\}$. Let $f_i = f|_{Be_i}$. Then $f = \oplus\sum_{i=1}^m f_i$ such that $f_i \in \mathrm{Iso}(Be_i, Be_{\pi(i)})$. Conversely, let $(f_1, f_2, ..., f_m)$ be a set of isomorphisms $f_i \in \mathrm{Iso}(Be_i, Be_{\pi(i)})$ for a permutation π of $\{1, 2, \ldots, m\}$. We claim that $f = \oplus\sum_{i=1}^m f_i \in \mathrm{Aut}_{B^G}(B)$. In fact, since $B = \oplus\sum_{i=1}^m Be_i$, for each $b \in B$, $f(b) = f(\oplus\sum_{i=1}^m be_i) = \oplus\sum_{i=1}^m f_i(be_i)$ where $f_i(be_i) \in Be_{\pi(i)}$. It is straightforward to verify that f preserves the addition and multiplication. Moreover, let $f(b) = 0$. Then $f_i(be_i) = 0$ for each i. But f_i is an isomorphism, so $be_i = 0$ for each i. Hence $b = 0$. Thus f is an injection. To show that f is a surjection, let $y \in B$. Then $y = \sum_{i=1}^m ye_{\pi(i)}$. By hypothesis, $f_i : Be_i \cong Be_{\pi(i)}$, so there exists $x_ie_i \in Be_i$ such that $f_i(x_ie_i) = ye_{\pi(i)}$ for each i. Let $x = \sum_{i=1}^m x_ie_i$. Then $xe_i = x_ie_i$ and $f(x) = f(\sum_{i=1}^m xe_i) = \sum_{i=1}^m f_i(xe_i) = \sum_{i=1}^m ye_{\pi(i)} = y$; and so f $(= \oplus\sum_{i=1}^m f_i)$ is a surjection. Moreover, for each $a \in B^G$, $f(a) = f(\sum_{i=1}^m ae_i) = \sum_{i=1}^m f_i(ae_i) = \sum_{i=1}^m ae_{\pi(i)} = a$. Thus $f = \oplus\sum_{i=1}^m f_i \in \mathrm{Aut}_{B^G}(B)$.

Combining Lemmas 4.2 and 4.4, we obtain a relation between $\mathrm{Aut}_{B^Ge_1}(Be_1)$ and $\mathrm{Aut}_{B^G}(B)$.

Theorem 4.5. *By keeping the notations of Lemma 4.4,*

$$Aut_{B^G}(B) = \oplus\sum_{i=1}^m \alpha_{i\pi}\alpha_i Aut_{B^Ge_1}(Be_1)\alpha_i^{-1}$$

where $\alpha_i \in \mathrm{Iso}(Be_1, Be_i)$, π *a permutation of* $\{1, 2, \ldots, m\}$, *and* $\alpha_{i\pi} \in \mathrm{Iso}(Be_i, Be_{\pi(i)})$.

In particular, for a semiconnected commutative Galois extension B with Galois group G, we derive a relation between $\mathrm{Aut}_{B^G}(B)$, $G(e_1)$, and the fat group of G.

Theorem 4.6. *Let B be a semiconnected commutative Galois extension with Galois group G. By keeping the notations of Lemma 4.4,*

$$Aut_{B^G}(B) = \oplus \sum_{i=1}^{m} g_{i\pi} g_i G(e_1) g_i^{-1} = \textit{the fat group of } G$$

where $g_i \in G(e_1, e_i)$, π a permutation of $\{1, 2, \ldots, m\}$, and $g_{i\pi} \in G(e_i, e_{\pi(i)})$.

Proof. Since Be_1 is a connected commutative Galois extension over $B^G e_1$ with Galois group $G(e_1)$, $G(e_1) = \mathrm{Aut}_{B^G e_1}(Be_1)$ ([2], Theorem 3.5). Thus $\mathrm{Aut}_{B^G}(B) = \oplus \sum_{i=1}^{m} g_{i\pi} g_i G(e_1) g_i^{-1}$ by Theorem 4.5 and Lemma 4.2. Moreover, for any $f \in \mathrm{Aut}_{B^G}(B)$, $f = \oplus \sum_{i=1}^{m} f_i$ such that $f_i \in g_{i\pi} g_i G(e_1) g_i^{-1} = g_{i\pi} G(e_i) \subset G|_{Be_i}$. Hence f is contained in the fat group of G. Thus $\mathrm{Aut}_{B^G}(B) =$ the fat group of G.

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