

ON SOME CORRESPONDENCE RELATIONS OF AN AZUMAYA ALGEBRA WITH A FINITE AUTOMORPHISM GROUP

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Abstract

Let B be an Azumaya algebra with a finite automorphism group G . Then a one-to-one correspondence is given between the set of commutative separable subalgebras of B and the set of the Azumaya automorphism extensions with subgroups of G in B . In particular, if B is an Azumaya automorphism extension with group G , a correspondence relation is also shown for separable subalgebras of B comparable with B^G .

1. Introduction

Let B be an Azumaya Galois extension or a DeMeyer-Kanzaki Galois extension with Galois group G . Then $B = B^G \cdot V_B(B^G) \cong B^G \otimes_{C^G} V_B(B^G)$ as C^G -algebras under the multiplication map where B^G is an Azumaya C^G -algebra, C the center of B , B^G and C^G the sets of elements in B and C respectively fixed under each element in G , and $V_B(B^G)$ is the commutator subring of B^G in B ([1], [2]). In [7], let B be a ring with a finite

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automorphism group G of B . We call B an Azumaya automorphism extension of B^G with group G if $B \cong B^G \otimes_{C^G} V_B(B^G)$ as Azumaya C^G -algebras under the multiplication map, and it was shown that there exists a one-to-one correspondence between the set of subgroups of G and a subset of the Azumaya automorphism subextensions of B ([7], Theorem 3.6 and 3.7). More correspondence relations between separable subextensions in some type of Galois extensions are given in [1], [2], [5], and [6]. Now, for an Azumaya algebra B with an automorphism group G (not necessarily Galois), in the present paper we shall show an equivalent condition under which the product $E \cdot V_B(E)$ of a commutator pair of separable subalgebras E and $V_B(E)$ is an Azumaya automorphism extension with a group induced by a subgroup of G . This derives a one-to-one correspondence between the set of the Azumaya automorphism extensions $E \cdot V_B(E)$ for a separable subalgebra E and the set of commutative separable subalgebras of B . Moreover, for an Azumaya automorphism extension B , we shall show a one-to-one correspondence relation for the set of separable subalgebras comparable with B^G . Using this correspondence, we can obtain an expression of the invariant subalgebra B^K for a subgroup K of G when B is an Azumaya automorphism and a Galois extension with Galois group G .

2. Definitions and Notations

Throughout, we employ all definitions and notations as given in [7]. B will represent a ring with 1, G a finite automorphism group of B , C the center of B , and B^G the set of elements in B fixed under each element in G . Let A be a subring of a ring B with the same identity 1. $V_B(A)$ denotes the commutator subring of A in B . We call B a separable extension of A if there exist $\{a_i, b_i$ in B , $i = 1, 2, \dots, m$ for some integer $m\}$ such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where \otimes is over A . An Azumaya algebra is a separable extension of its center. B is called a Galois extension of B^G with Galois group G if there exist elements $\{a_i, b_i$ in B , $i = 1, 2, \dots, m\}$ for some integer m such that $\sum_{i=1}^m a_i g(b_i) = \delta_{1,g}$ for each $g \in G$. Such a set $\{a_i, b_i\}$ is called a G -Galois system for

B . B is called an Azumaya Galois extension if B is a Galois extension with Galois group G over an Azumaya C^G -algebra B^G ([1]), and B a DeMeyer-Kanzaki Galois extension of B^G with Galois group G if B is an Azumaya C -algebra and C is a Galois algebra with Galois group $G|_C \cong G$ ([2]). We call B an Azumaya automorphism extension of B^G with group G if $B \cong B^G \otimes_{C^G} V_B(B^G)$ as Azumaya C^G -algebras under the multiplication map ([7]).

3. Commutative Separable Subalgebras

Let B be an Azumaya algebra with a finite automorphism group G and E a separable subalgebra such that $E \subset B^K$ for some subgroup K of G . We shall give an equivalent condition under which $E \cdot V_B(E)$ is an Azumaya automorphism extension with group K' induced by K . Then the set of such Azumaya automorphism extensions $E \cdot V_B(E)$ with a group K' is shown to be in a one-to-one correspondence with the set of commutative separable subalgebras of B .

Lemma 3.1. *Let B be an Azumaya C -algebra and E a separable subalgebra of B with center S . Then $E \cdot V_B(E) = V_B(S)$ as Azumaya S -algebras.*

Proof. Since E is a separable subalgebra of B , $V_B(E)$ is a separable subalgebra of B such that $V_B(V_B(E)) = E$ by the commutator theorem for Azumaya algebras ([3], Theorem 4.3, page 57); and so E and $V_B(E)$ are Azumaya S -algebras. Also, by noting that S is a separable subalgebra of B , $V_B(V_B(S)) = S$. This implies that $V_B(S)$ is an Azumaya S -algebra. Clearly, E and $V_B(E)$ are contained in $V_B(S)$. Hence $V_B(E) = V_{V_B(S)}(E)$ and $V_B(S) = E \cdot V_B(E)$ by the commutator theorem for Azumaya algebras.

Theorem 3.2. *Let B be an Azumaya C -algebra with a finite automorphism group G , E a separable subalgebra such that $E \subset B^K$ for some subgroup K of G , and S the center of E . Then*

(1) $E \cdot V_B(E)$ is invariant under K , and

(2) $E \cdot V_B(E)$ is an S -Azumaya automorphism extension with group K' induced by K if and only if $(E \cdot V_B(E))^K$ is an Azumaya S -algebra.

Proof. (1) Since $E \subset B^K$, $k(V_B(E)) = V_B(E)$ for each $k \in K$; and so $k(E \cdot V_B(E)) = E \cdot V_B(E)$ for each $k \in K$.

(2) (\implies) Denote $E \cdot V_B(E)$ by Δ . Since Δ is an S -Azumaya automorphism extension with group K' induced K , $\Delta \cong \Delta^K \otimes_S V_\Delta(\Delta^K)$ as Azumaya S -algebras where S is the center of Δ and Δ^K ; and so $(E \cdot V_B(E))^K (= \Delta^K)$ is an Azumaya S -algebra.

(\impliedby) Since E is a separable C -subalgebra, S , the center of E , is separable over C ([3], Theorem 3.8, page 55). By hypothesis, Δ^K is an Azumaya S -algebra, so Δ^K is a separable C -subalgebra of B . Hence, by Lemma 3.1, $E \cdot V_B(E) = V_B(S) = \Delta^K \cdot V_B(\Delta^K)$ as Azumaya S -algebras. Moreover, since $\Delta = E \cdot V_B(E) = \Delta^K \cdot V_B(\Delta^K)$, $V_B(\Delta^K) \subset \Delta$; and so $V_B(\Delta^K) = V_\Delta(\Delta^K)$. Thus, $\Delta = \Delta^K \cdot V_B(\Delta^K) = \Delta^K \cdot V_\Delta(\Delta^K)$ as Azumaya S -algebras. This implies that Δ is an Azumaya automorphism extension with group induced by K .

Remark. By Lemma 3.1, Theorem 3.2 can also be proved by Theorem 3.1 in [7].

Let S be a commutative separable subalgebra of B and $\mathcal{C}_S = \{E \mid E \text{ is a separable subalgebra of } B \text{ contained in } B^K \text{ for some subgroup } K \text{ of } G \text{ with center } S \text{ and } E \cdot V_B(E) \text{ is an Azumaya automorphism subextension of } B \text{ with group } K' \text{ induced by } K\}$. We note that, for any $E \in \mathcal{C}_S$, $E \cdot V_B(E) = V_B(S)$ as Azumaya S -algebras by Lemma 3.1.

Theorem 3.3. *Let B be an Azumaya C -algebra with a finite automorphism group G . There exists a one-to-one correspondence between the set of Azumaya automorphism subextensions $E \cdot V_B(E)$ with group K' induced by some subgroup K of G and the set of commutative separable subalgebras of B .*

Proof. Let $\alpha : E \cdot V_B(E) \longrightarrow S$ where S is the center of $E \in \mathcal{C}_S$. Then α is well defined. Also, let $\alpha(E \cdot V_B(E)) = \alpha(F \cdot V_B(F)) = S$. Then $E \cdot V_B(E) = V_B(S) = F \cdot V_B(F)$ by Lemma 3.1. Thus, α is one-to-one. Next, we show that α is onto. Let S be a commutative separable subalgebra of B . Since B is an Azumaya C -algebra and S is a separable subalgebra of B , $V_B(V_B(S)) = S$. Hence $V_B(S)$ is an Azumaya S -algebra. Let $K = \{g \in G \mid g(s) = s \text{ for each } s \in S\}$. Then $V_B(S)$ is invariant under K , that is, $k(V_B(S)) = V_B(S)$ for all $k \in K$. Thus, $S \cdot V_B(S)$ is an Azumaya automorphism extension in B with group K' induced by K and $\alpha(S \cdot V_B(S)) = S$.

4. Separable Subalgebras

Let B be an Azumaya automorphism extension with group G . We shall show a one-to-one correspondence relation for the separable subalgebras of B comparable with B^G . This derives an expression of the invariant subalgebra B^K for a subgroup K of G when B is also a Galois extension with Galois group G . We begin with some properties of an Azumaya algebra.

Lemma 4.1. *Let B be an Azumaya C -algebra and A an Azumaya subalgebra of B . Then for any separable subalgebra E of B containing A , $E = A \cdot (E \cap V_B(A))$ such that $E \cap V_B(A)$ is a separable subalgebra of B .*

Proof. Since A is an Azumaya C -algebra and E is a A -bimodule, $E \cong A \otimes_C V_E(A)$ under the multiplication map ([3], Corollary 3.6, page 54). It is easy to check that this is also an algebra isomorphism. Moreover, $V_E(A) = E \cap V_B(A)$, so it suffices to show that $V_E(A)$ is a separable subalgebra of B . In fact, E is a separable extension of A because it is a separable extension of C and $C \subset A \subset E$. Noting that C is a direct summand of A as a C -bimodule, $V_E(A)$ is a separable C -algebra ([4], Corollary 2.12).

Let A be an Azumaya subalgebra of B as given in Lemma 4.1; then $B = A \otimes_C V_B(A)$ as Azumaya algebras where $V_B(A)$ is also an Azumaya subalgebra of B such that $V_B(V_B(A)) = A$ ([3], Theorem 4.3, page 57). Hence, by the same argument as given in the proof of Lemma 4.1, we have an expression of a separable subalgebra of B containing $V_B(A)$, that is, for any separable subalgebra F of B containing $V_B(A)$, $F = V_B(A) \cdot (F \cap A)$ such that $F \cap A$ is a separable subalgebra of B .

Lemma 4.2. *Let B be an Azumaya C -algebra and A an Azumaya subalgebra of B . Then $\alpha : E \rightarrow E \cap V_B(A)$ is a bijection between the set of separable subalgebras of B containing A and the set of separable subalgebras of B contained in $V_B(A)$ with the inverse map $\alpha^{-1} : E_0 \rightarrow A \cdot E_0$.*

Proof. By Lemma 4.1, α is well defined. Let E and E' be separable subalgebras of B containing A such that $\alpha(E) = \alpha(E')$. Then, $E \cap V_B(A) = E' \cap V_B(A)$. Hence, by Lemma 4.1, $E = A \cdot (E \cap V_B(A)) = A \cdot (E' \cap V_B(A)) = E'$. This implies that α is one-to-one. Next, we claim that α is onto. For any separable subalgebra E_0 of B contained in $V_B(A)$, $A \cdot E_0 (\cong A \otimes_C E_0)$ is a separable subalgebra of B containing A . Hence $V_B(V_B(A \cdot E_0)) = A \cdot E_0$ by the commutator theorem for Azumaya algebras. Therefore, $\alpha(A \cdot E_0) = (A \cdot E_0) \cap V_B(A) = V_B(V_B(A \cdot E_0)) \cap V_B(A) = V_B(V_{V_B(A)}(E_0)) \cap V_B(A) = V_{V_B(A)}(V_{V_B(A)}(E_0))$. Noting that $V_B(A)$ is an Azumaya C -algebra and E_0 is separable subalgebra of $V_B(A)$, we have $\alpha(A \cdot E_0) = V_{V_B(A)}(V_{V_B(A)}(E_0)) = E_0$. This completes the proof.

Lemma 4.3. *Let B be an Azumaya C -algebra and A an Azumaya subalgebra of B . Then $\beta : F \rightarrow F \cap A$ is a bijection between the set of separable subalgebras of B containing $V_B(A)$ and the set of separable subalgebras of B contained in A with the inverse map $\beta^{-1} : F_0 \rightarrow V_B(A) \cdot F_0$.*

Proof. It is similar to the proof of Lemma 4.2.

Applying Lemma 4.2 and Lemma 4.3 to an Azumaya automorphism extension B , we have the following correspondence theorem for the separable subalgebras of B comparable with B^G .

Theorem 4.4. *Let B be an Azumaya automorphism extension with group G . Then (1) $\alpha : E \rightarrow E \cap V_B(B^G)$ is a bijection between the set of separable subalgebras of B containing B^G and the set of separable subalgebras of B contained in $V_B(B^G)$ with the inverse map $\alpha^{-1} : E_0 \rightarrow B^G \cdot E_0$, and (2) $\beta : F \rightarrow F \cap B^G$ is a bijection between the set of separable subalgebras of B containing $V_B(B^G)$ and the set of separable subalgebras of B contained in B^G with the inverse map $\beta^{-1} : F_0 \rightarrow V_B(B^G) \cdot F_0$.*

Proof. Since B is an Azumaya automorphism extension, B^G is an Azumaya subalgebra of B . Hence, by letting $A = B^G$, Theorem 4.4 is an immediate consequence of Lemma 4.2 and Lemma 4.3.

Now for an Azumaya automorphism and a Galois extension B with Galois group G , we can derive an expression of the invariant subring B^K for a subgroup K of G .

Theorem 4.5. *Let B be an Azumaya automorphism and a Galois extension B with Galois group G of order $|G|$ invertible in B . Then, for a subgroup K of G , $B^K = B^G \cdot (V_B(B^G))^K$.*

Proof. We first show that B^K is a separable subalgebra of B containing B^G . Since B is a Galois extension of B^G with Galois group G , B is also a Galois extension of B^K with Galois group K . Hence B is a finitely generated and projective left (or right) B^K -module. Moreover, noting that $|G|^{-1} \in B$, we have $|K|^{-1} \in B$, and so B^K is a direct

summand of the Azumaya C -algebra B as a B^K -bimodule. Thus, B^K is a separable algebra over C by the proof of Theorem 3.8 on page 55 in [3]. Therefore, by Lemma 4.1, $B^K = B^G \cdot (B^K \cap V_B(B^G)) = B^G \cdot V_{B^K}(B^G) = B^G \cdot (V_B(B^G))^K$.

We conclude the present paper with an example of an Azumaya automorphism and Galois extension B with the expression of B^K as given in Theorem 4.5.

Let $Q[i, j, k]$ be the quaternion algebra over the rational field Q , $B = M_2(Q[i, j, k])$

the 2×2 matrix ring over $Q[i, j, k]$, and $G = \{1, g_i, g_j, g_k\}$ where

$$g_i \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} iai^{-1} & ibi^{-1} \\ ici^{-1} & idi^{-1} \end{pmatrix}, \quad g_j \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} jaj^{-1} & jbj^{-1} \\ jcj^{-1} & jdj^{-1} \end{pmatrix}, \text{ and}$$

$$g_k \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} kaka^{-1} & kbk^{-1} \\ kck^{-1} & kdk^{-1} \end{pmatrix} \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B. \text{ Then,}$$

- (1) The center of B is $C = \left\{ \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \mid q \in Q \right\} \cong Q$, and $C^G = C$.
- (2) $B^G = M_2(Q)$, the 2×2 matrix ring over Q . Hence B^G is an Azumaya Q -algebra.
- (3) $V_B(B^G) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in Q[i, j, k] \right\} \cong Q[i, j, k]$ which is a Galois extension of Q with Galois group induced by and isomorphic with G where $\{\frac{1}{2}, \frac{1}{2}i, \frac{1}{2}j, \frac{1}{2}k; \frac{1}{2}, -\frac{1}{2}i, -\frac{1}{2}j, -\frac{1}{2}k\}$ is a G -Galois system.
- (4) $B \cong B^G \otimes_Q V_B(B^G)$ as Azumaya Q -algebras under the multiplication map.
- (5) By (3), B is a Galois extension with Galois group G .
- (6) B is an Azumaya automorphism and a Galois extension of B^G with Galois group G by (4) and (5).
- (7) The nontrivial subgroups of G are $K_i = \{1, g_i\}$, $K_j = \{1, g_j\}$, and $K_k = \{1, g_k\}$.
- (8) $B^{K_i} = M_2(Q[i]) = M_2(Q) \cdot CQ[i] = B^G \cdot (V_B(B^G))^{K_i}$,
 $B^{K_j} = M_2(Q[j]) = M_2(Q) \cdot CQ[j] = B^G \cdot (V_B(B^G))^{K_j}$, and
 $B^{K_k} = M_2(Q[k]) = M_2(Q) \cdot CQ[k] = B^G \cdot (V_B(B^G))^{K_k}$.
- (9) Some of the separable subalgebras of B containing B^G are

$$\begin{aligned}
B^G &= M_2(Q) = M_2(Q) \cdot CQ = B^G \cdot (B^G \cap V_B(B^G)), \\
E_i &= M_2(Q[i]) = M_2(Q) \cdot CQ[i] = B^G \cdot (E_i \cap V_B(B^G)), \\
E_j &= M_2(Q[j]) = M_2(Q) \cdot CQ[j] = B^G \cdot (E_j \cap V_B(B^G)), \\
E_k &= M_2(Q[k]) = M_2(Q) \cdot CQ[k] = B^G \cdot (E_k \cap V_B(B^G)), \text{ and} \\
B &= M_2(Q[i, j, k]) = M_2(Q) \cdot CQ[i, j, k] = B^G \cdot (B \cap V_B(B^G)).
\end{aligned}$$

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