A GENERALIZATION OF THE DEMEYER THEOREM FOR CENTRAL GALOIS ALGEBRAS

GEORGE SZETO

Department of Mathematics, Bradley University
Peoria, Illinois 61625 – U.S.A.
E-mail: szeto@bradley.edu

LIANYONG XUE

Department of Mathematics, Bradley University
Peoria, Illinois 61625 – U.S.A.
E-mail: lxue@bradley.edu

Let A be an Azumaya algebra over a semi-local ring R, and M and N finitely generated projective left A-modules such that $\text{rank}_M = \text{rank}_N$. Then $M \cong N$. Thus it can be shown that a central Galois algebra over R is a projective group algebra, and a Galois algebra is a direct sum of projective group algebras.

1. Introduction

Let A be an Azumaya algebra over a semi-local ring R with no idempotents but 0 and 1, and M and N indecomposable finitely generated projective left A-modules. Then it was shown that $M \cong N$ ([3], Theorem 1). Thus the Noether-Skolem theorem can be generalized from central simple algebras to Azumaya algebras over a semi-local ring with no idempotents but 0 and 1, that is, any automorphism of A is inner ([1], page 122). Consequently, any central Galois algebra over a semi-local ring with no idempotents but 0 and 1 is a projective group algebra ([1], Theorem 6). The purpose of the present paper is to generalize the above result to an Azumaya algebra A over a semi-local ring R (not necessarily with no idempotents but 0 and 1). Let M and N be finitely generated projective left A-modules. If the rank functions of M and N over R are equal, then $M \cong N$, where $\text{rank}_M(p) =$ the rank of the free R_p-module M_p over the local ring R_p at the prime ideal p of R. Then we shall show that the Noether-Skolem theorem holds for A, and a central Galois algebra over R with Galois group G is a projective group algebra of G over R, RG_f, with a factor set $f : G \times G \longrightarrow \{\text{units of } R\}$ as defined by F. R. DeMeyer in [1]. Thus a Galois algebra (not necessarily central) over R can be shown to be a direct sum of projective group algebras.

2. Basic Definitions and Notations

Throughout this paper, B will represent a ring with 1, G a finite automorphism group of B, C the center of B, and B^G the set of elements in B fixed under each element in G.

Let A be a subring of a ring B with the same identity 1. We call B a separable extension of A if there exist $\{a_i, b_i$ in B, $i = 1, 2, ..., m$ for some integer $m\}$ such that $\sum a_i b_i = 1$, and $\sum ba_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where $\otimes$ is over A. An Azumaya algebra is a separable extension of its center. A ring B is called a Galois extension of B^G with Galois group G if there exist elements $\{a_i, b_i$ in B, $i = 1, 2, ..., m\}$ for some integer m such that $\sum_{i=1}^{m} a_i g(b_i) = \delta_{1,g}$ for each $g \in G$, a Galois algebra over R if B is a Galois extension of R which is contained in C, and a central Galois extension if B is a Galois extension over its center C.

Let P be a projective module over a commutative ring R. Then for a prime ideal p of R, $P_p (= P \otimes_R R_p)$ is a free module over $R_p (=$ the local ring of R at $p)$, and the rank of P_p over R_p is the number of copies of R_p in P_p. We denote the rank function associated with P from the prime spectrum of R to nonnegative integers by rank_P, that is, $\text{rank}_P(p) =$ the number of copies of R_p in P_p.

3. Galois Extensions

Let R be a commutative ring with 1, M a finitely generated projective R-module. We recall that the rank function associated with M from the prime spectrum of R to nonnegative integers is denoted by rank_M. Let A be an Azumaya algebra over a semi-local ring R. We shall characterize a finitely generated projective left A-module M in terms of rank_M. This derives the Noether-Skolem theorem for A. Consequently, it can be shown that any central Galois algebra over R is a projective group algebra, and a Galois algebra over R is a direct sum of projective group algebras where a projective group algebra is defined by F. R. DeMeyer in [1]. We begin with a classification of finitely generated and projective modules over an Azumaya algebra by the rank function.

Lemma 3.1. *Let M and N be finitely generated projective modules over a semi-local ring R. If $\text{rank}_M = \text{rank}_N = k$ for some integer k, then $M \cong N \cong F_k$ which is a free R-module of rank k.*

Proof. Since R is semi-local, there are minimal idempotents $\{e_i \,|\, i = 1, 2, ..., m$ for some integer $m\}$ summing to 1. Hence Re_i is a semi-local ring with no idempotents but 0 and e_i such that $\text{rank}_{Me_i} = \text{rank}_{Ne_i} = k$ for each i. Let J be the Jacobson radical of Re_i. Then $Me_i/JMe_i \cong Ne_i/JNe_i$. Thus $Me_i \cong Ne_i \cong F_k e_i$ by using the Nakayama Lemma. This implies that $M \cong N \cong F_k$.

Theorem 3.2. *Let A be an Azumaya algebra over a semi-local ring R, and M and N finitely generated projective left A-modules. If $rank_M = rank_N = k$ for some integer k, then $M \cong N$ as left A-modules.*

Proof. Let $\{e_i \mid i = 1, 2, ..., m$ for some integer $m\}$ be the set of minimal idempotents in R summing to 1. We claim that $Me_i \cong Ne_i$ for each i. In fact, Let J be the Jacobson radical of Re_i. Noting that $\text{rank}_{Me_i} = \text{rank}_{Ne_i} = k$ (for $\text{Spec}(R) = \cup_{i=1}^{m}\text{Spec}(Re_i)$), we have that $Me_i \cong Ne_i \cong F_k e_i$ by Lemma 3.1. Thus $Me_i/JMe_i \cong Ne_i/JNe_i$ as left Ae_i/JAe_i-modules (for Ae_i/JAe_i is a direct sum of central simple algebras). Let $\pi : Me_i \longrightarrow Ne_i/JNe_i\,(\cong Me_i/JMe_i)$ be the surjection homomorphism. Since Ne_i is a finitely generated projective left Ae_i-module such that $Ne_i \longrightarrow Ne_i/JNe_i$ is surjective, there exists a homomorphism $\alpha : Ne_i \longrightarrow Me_i$ such that $Me_i = \alpha(Ne_i) + JMe_i$. But then $Me_i = \alpha(Ne_i)$ by the Nakayama Lemma. This implies that α is a surjection. Let $K = \ker(\alpha)$. Then $0 \longrightarrow K \longrightarrow Ne_i \longrightarrow Me_i \longrightarrow 0$ is a split exact sequence. Since Me_i is a finitely generated projective left Ae_i-module, $Ne_i \cong Me_i \oplus K$. But $\text{rank}_{Me_i} = \text{rank}_{Ne_i}$, so $K_p = 0$ for each $p \in \text{Spec}(Re_i)$. Thus $K = 0$. Therefore $Ne_i \cong Me_i$; and so $N \cong M$.

As a consequence of Theorem 3.2, we have a classification of finitely generated projective left A-modules.

Corollary 3.3. *Let A be an Azumaya algebra over a semi-local ring R, and M and N finitely generated projective left A-modules. If $rank_M = rank_N$, then $M \cong N$.*

Proof. Let Q be a finitely generated projective left A-module. Then Q is a finitely generated projective left R-module (for A is an Azumaya algebra over R). Noting that Re_i is a semi-local ring with no idempotents but 0 and e_i, $\text{rank}_{Me_i} = \text{rank}_{Ne_i} = k_i$ for some integer k_i for each i ([3], Theorem 1). Moreover Ae_i is an Azumaya algebra over the semi-local ring Re_i, we have that $Me_i \cong Ne_i$ for each i by Theorem 3.2. Thus $N \cong M$.

Now we show that the Noether-Skolem theorem for Azumaya algebras over a semi-local ring.

Theorem 3.4. *Let A be an Azumaya algebra over a semi-local ring R. If α is an automorphism of A, then α is an inner automorphism.*

Proof. Let A^o be the opposite algebra of A and $A^e = A \otimes_R A^o$. Then A is a left A^e-module by $(x \otimes y)(a) = xay$ for each $x \otimes y \in A^e$ and $a \in A$, which is denoted by

A_1. Also, A is a left A^e-module by $(x \otimes y)(a) = \alpha(x)ay$ for each $x \otimes y \in A^e$ and $a \in A$, which is denoted by A_2. Noting that A^e is an Azumaya R-algebra (for A is an Azumaya R-algebra) and that both A_1 and A_2 are finitely generated projective left A^e-modules ([4], Proposition 1.1, page 40) such that $\text{rank}_{A_1} = \text{rank}_{A_2}$, we have that $\pi : A_1 \cong A_2$ as left A^e-modules by Corollary 3.3. Thus for each $a \in A$, $\pi(a) = \pi((a \otimes 1) \cdot 1) = \pi((1 \otimes a) \cdot 1)$, that is, $(a \otimes 1) \cdot \pi(1) = (1 \otimes a) \cdot \pi(1)$. This implies that $\alpha(a) \cdot \pi(1) = \pi(1) \cdot a$. Moreover, since $\pi : A_1 \cong A_2$, there exists an element $b \in A_1$ such that $\pi(b) = \pi(1) \cdot b = 1 = \alpha(b) \cdot \pi(1)$. Thus $\pi(1)$ is a unit in A such that $\alpha(a) = (\pi(1))a(\pi(1))^{-1}$ for each $a \in A$. This implies that α is an inner automorphism of A.

As an application of Theorem 3.4, the structure of a central Galois algebra over a semi-local ring can be derived. As defined by F. R. DeMeyer ([3]), RG_f is called a projective group algebra of a finite group G over a commutative ring R with a factor set $f : G \times G \longrightarrow \{\text{units of } R\}$ if RG_f is a free R-module with a basis $\{x_i \,|\, g_i \in G,\ i = 1, 2, ..., m$ for some integer $m\}$ such that $rx_i = x_i r$ for each $r \in R$ and $x_i x_j = x_k \cdot f(g_i, g_j)$ where $g_i g_j = g_k$ for $g_i, g_j \in G$.

Corollary 3.5. *If A is a central Galois algebra over a semi-local ring R with Galois group G, then A is isomorphic with a projective group algebra RG_f with a factor set $f : G \times G \longrightarrow$ the units of R.*

Proof. By Theorem 3.4, G is an inner Galois group of A, so $A \cong RG_f$ ([1], Theorem 6).

By Theorem 3.4, we have the following classes of Galois algebras (not necessarily central) which are also projective group algebras. Thus Theorem 6 in [1] is generalized to Galois algebras over a semi-local ring.

Theorem 3.6. *If B is a Galois algebra with Galois group G over a semi-local ring R with no idempotents but 0 and 1, then B is a projective group algebra.*

Proof. Let C be the center of B and $H = \{g \in G \,|\, g(c) = c$ for each $c \in C\}$. Then B is a central Galois algebra with Galois group H ([2], Theorem 1). Moreover, since R is semi-local, C is a semi-local ring. Hence H is inner by Theorem 3.4; and so $B = CH_f$ which is a projective group algebra ([1], Theorem 6).

Theorem 3.7. *Let B be a Galois algebra over a semi-local ring R with Galois group G, C the center of B, $H = \{g \in G \,|\, g(c) = c$ for each $c \in C\}$, and $J_g = \{a \in B \,|\, ax = g(x)a$ for every $x \in B\}$. If $J_g = \{0\}$ for each $g \notin H$, then B is a projective group algebra.*

Proof. Since $J_g = \{0\}$ for each $g \notin H$, by Proposition 3 in [5], B is a central Galois algebra with Galois group H. Noting that C is a semi-local ring and that H is inner by Theorem 3.4, we have that B is a projective group algebra ([1], Theorem 6).

In general, for any Galois algebra over a semi-local ring, we shall show that B is a direct sum of projective group algebras. The following lemma for a Galois extension with finitely many central idempotents plays an important role.

Lemma 3.8. *Let B be a Galois extension of B^G with Galois group G. If B contains only finitely many central idempotents, then for any minimal central idempotent e, $(Be)^{G(e)} = B^G e$ where $G(e) = \{g \in G \,|\, g(e) = e\}$.*

Proof. Since e is minimal, $e \cdot g(e) = e$ or 0 for any $g \in G$. Thus $(Be)^{G(e)} = B^G e$ ([6], Lemma 9).

Theorem 3.9. *Let B be a Galois algebra over a semi-local ring R with Galois group G. If $G(e_i) \neq \{1\}$ for each minimal central idempotent, then B is a direct sum of projective group algebras.*

Proof. Let C be the center of B. Since B is a Galois algebra over a semi-local ring R, C is also a semi-local ring. Hence B has only finitely many central idempotents. Let e be a minimal central idempotent. Then Be is a Galois extension of $(Be)^{G(e)}$ with Galois group $G(e)$ where $G(e) = \{g \in G \,|\, g(e) = e\}$ ([7], Lemma 3.7). By Lemma 3.8, $(Be)^{G(e)} = B^G e = Re$, so Be is a Galois algebra over Re with Galois group $G(e)$. Noting that Re is a semi-local ring with no idempotents but 0 and e, we conclude that Be is a projective group algebra by Theorem 3.6. But B contains only finitely many central idempotents, so $B = \oplus \sum_{i=1}^{m} Be_i$ where $\{e_i \,|\, i = 1, 2, ..., m$ for some integer $m\}$ are all minimal central idempotents of B. Therefore B is a direct sum of projective group algebras.

We note that the condition in Theorem 3.9, $G(e_i) \neq \{1\}$, is important to have a nontrivial Galois algebra Be_i over Re_i. In case $G(e_i) = \{1\}$ for some i, we shall employ the structure theorem as given in [7] for B to avoid this situation.

Theorem 3.10. *If B is a Galois algebra over a semi-local ring R with Galois group G, then $B = A \oplus B'$ where A is a commutative Galois algebra with Galois group $G|_A \cong G$ and B' is a direct sum of projective group algebras.*

Proof. By Theorem 3.8 in [7], there exist central idempotents $\{E_j \,|\, j = 1, 2, ..., n$ for some integer $n\}$ such that $B = BE_0 \oplus (\oplus \sum_{j=1}^{n} BE_j)$ where BE_j is a central Galois algebra over CE_j with Galois group H_j contained in G for each $j = 1, 2, ..., n$ and BE_0 is a commutative Galois algebra over RE_0 with Galois group $G|_{BE_0} \cong G$. Since RE_j is a semi-local ring, CE_j is a semi-local ring; and so BE_j is a projective group algebra for each $j = 1, 2, ..., n$ by Theorem 3.7.

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