

The Commutator Hopf Galois Extensions

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ABSTRACT. Let H be a finite dimensional Hopf algebra over a field k and H^* the dual Hopf algebra of H . Then a commutator right H^* -Galois extension B of B^H is characterized in terms of the smash product $B\#H$ and some relationships between such a B and the Hopf Galois Azumaya or Hopf Galois Hirata extensions are also given.

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1 Introduction.

Let H be a finite dimensional Hopf algebra over a field k , H^* the dual Hopf algebra of H , B a left module algebra over H with center C , and $V_B(B^H)$ the commutator subring of B^H in B . Let B be a right H^* -Galois extension of B^H , in [3], it was shown that $V_B(B^H)$ is a right H^* -Galois extension of Z where Z is the center of B^H if and only if $B \cong B^H \otimes_Z V_B(B^H)$ and $V_B(B^H)$ is a finitely generated and projective Z -module ([3], Lemma 2.8). Now we call B a commutator Hopf Galois extension of B^H if $V_B(B^H)$ is a right H^* -Galois extension of Z . When $H = (kG)^*$ for a finite automorphism group G of B , a commutator Hopf Galois extension is a commutator Galois extension as studied in [6]. The purpose of the present paper is to characterize a commutator Hopf Galois extension in terms of the Hirata separability of the smash product $B\#H$ over B^H . Moreover, a relationship is also given between the class of commutator Hopf Galois extensions and the

class of Hopf Galois Azumaya extensions as studied in [4] and [7] or the class of Hopf Galois Hirata extensions, respectively, where B is called a right H^* -Galois Hirata extension of B^H if B is a right H^* -Galois extension and a Hirata separable extension of B^H as given in [8].

2 Basic Definitions and Notations.

Throughout, H denotes a finite dimensional Hopf algebra over a field k with comultiplication Δ and counit ε , H^* the dual Hopf algebra of H , B a left H -module algebra, C the center of B , $B^H = \{b \in B \mid hb = \varepsilon(h)b \text{ for all } h \in H\}$ which is called the H -invariants of B , and $B\#H$ the smash product of B with H where $B\#H = B \otimes_k H$ such that for all $b\#h$ and $b'\#h'$ in $B\#H$, $(b\#h)(b'\#h') = \sum b(h_1b')\#h_2h'$ where $\Delta(h) = \sum h_1 \otimes h_2$. B is called a right H^* -Galois extension of B^H if B is a right H^* -comodule algebra with structure map $\rho : B \rightarrow B \otimes_k H^*$ such that $\beta : B \otimes_{B^H} B \rightarrow B \otimes_k H^*$ is a bijection where $\beta(a \otimes b) = (a \otimes 1)\rho(b)$.

For a subring A of B with the same identity 1, we denote the commutator subring of A in B by $V_B(A)$. we call B a separable extension of A if there exist $\{a_i, b_i \text{ in } B, i = 1, 2, \dots, m \text{ for some integer } m\}$ such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where \otimes is over A . An Azumaya algebra is a separable extension of its center. A ring B is called a Hirata separable extension of A if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B -bimodule. A right H^* -Galois extension B is called a commutator right H^* -Galois extension if $V_B(B^H)$ is a right H^* -Galois extension of the center of B^H , a right H^* -Galois Azumaya extension if B is separable over B^H which is an Azumaya algebra over C^H , and a right H^* -Galois Hirata extension if B is also a Hirata separable extension of B^H . Throughout, an H^* -Galois extension means a right H^* -Galois extension unless it is stated otherwise.

3. The Commutator Hopf Galois Extensions.

In this section, we shall characterize a commutator Hopf Galois extension in terms of the smash product $B\#H$.

Theorem 3.1. *Let B be an H^* -Galois extension of B^H and Z the center of B^H .*

Then the following statements are equivalent:

- (1) B is a commutator H^* -Galois extension of B^H ,
- (2) $B\#H$ is a projective Hirata separable extension of B^H and contains B^H as a direct summand as a B^H -bimodule,
- (3) $B\#H \cong B^H \otimes_Z (V_B(B^H)\#H)$ and $V_B(B^H)\#H$ is a finitely generated and projective Z -module, and
- (4) $V_B(B^H)\#H$ is an Azumaya Z -algebra.

Proof. (1) \implies (4) Since B is a commutator H^* -Galois extension of B^H , $V_B(B^H)$ is an H^* -Galois extension of Z . Hence $V_B(B^H)\#H \cong \text{Hom}_Z(V_B(B^H), V_B(B^H))$ such that $V_B(B^H)$ is a finitely generated and projective Z -module ([3], Theorem 1.7). But Z is commutative with 1, so $V_B(B^H)$ is a progenerator of Z . Thus $\text{Hom}_Z(V_B(B^H), V_B(B^H))$ is an Azumaya Z -algebra ([1], Proposition 4.1 on page 56); and so $V_B(B^H)\#H$ is an Azumaya Z -algebra.

(4) \implies (1) Since $V_B(B^H)\#H$ is an Azumaya Z -algebra, finitely generated and projective over $V_B(B^H)$, $V_B(B^H)\#H$ is a Hirata separable extension of $V_B(B^H)$ ([2], Theorem 1). But $V_B(B^H)$ is a progenerator of $V_B(B^H)$, so $V_B(B^H)$ is a progenerator of $V_B(B^H)\#H$ ([2], Lemma). Hence $V_B(B^H)$ is an H^* -Galois extension of $(V_B(B^H))^H$ ([4], Theorem 2.2). Noting that $(V_B(B^H))^H = V_{B^H}(B^H) = Z$, we have that B is a commutator H^* -Galois extension of B^H .

(1) \implies (2) By hypothesis, $V_B(B^H)$ is an H^* -Galois extension of Z , so $B \cong B^H \otimes_Z V_B(B^H)$ such that $V_B(B^H)$ is a finitely generated and projective Z -module ([3], Lemma 2.8). Noting that the above isomorphism is also a ring isomorphism, we have that $B\#H \cong$

$(B^H \otimes_Z V_B(B^H))\#H \cong B^H \otimes_Z (V_B(B^H)\#H)$. But by (1) \implies (4), $V_B(B^H)\#H$ is an Azumaya Z -algebra, so $B^H \otimes_Z (V_B(B^H)\#H)$ is a projective Hirata separable extension of B^H . Also Z is a direct summand of $V_B(B^H)\#H$ as a Z -bimodule ([1], Lemma 3.1, page 51), so B^H is a direct summand of $B^H \otimes_Z (V_B(B^H)\#H)$ as a B^H -bimodule. Thus (2) holds.

(2) \implies (4) Since $B\#H$ is a projective Hirata separable extension of B^H and contains B^H as a direct summand as a B^H -bimodule, $V_{B\#H}(B^H)$ is a separable Z -algebra such that $V_{B\#H}(V_{B\#H}(B^H)) = B^H$ ([5], Proposition 1.3 and Proposition 1.4). This implies that the center of $B\#H$, B^H , and $V_{B\#H}(B^H)$ are the same Z . Thus $V_B(B^H)\#H (= V_{B\#H}(B^H))$ is an Azumaya Z -algebra.

(1) \implies (3) Since $V_B(B^H)$ is an H^* -Galois extension of Z , $B \cong B^H \otimes_Z V_B(B^H)$ such that $V_B(B^H)$ is a finitely generated and projective Z -module ([3], Lemma 2.8). Noting that this isomorphism is also a ring isomorphism, we have that $B\#H \cong (B^H \otimes_Z V_B(B^H))\#H \cong B^H \otimes_Z (V_B(B^H)\#H)$ such that $V_B(B^H)\#H$ is a finitely generated and projective Z -module by (1) \implies (4).

(3) \implies (1) Since H is a finite dimensional Hopf algebra over a field k , the statement (3) implies that $B \cong B^H \otimes_Z V_B(B^H)$ and $V_B(B^H)$ is a finitely generated and projective Z -module. Thus $V_B(B^H)$ is an H^* -Galois extension of Z ([3], Lemma 2.8).

4. Three Kinds of Hopf Galois Extensions.

In section 2, we call B an H^* -Galois Hirata extension of B^H if B is an H^* -Galois extension and a Hirata separable extension of B^H . In [8], it was shown that an H^* -Galois extension B of B^H is a Hirata separable extension of B^H if and only if $V_B(B^H)$ is a left H -Galois extension of C ([8], Theorem 2.6). In this section, we shall show that an H^* -Galois Hirata extension of B^H equal to B^{H^*} is a commutator H^* -Galois extension of B^H . Moreover, an equivalent condition is given under which a commutator H^* -Galois extension

of B^H becomes an H^* -Galois Hirata extension of B^H or an H^* -Galois Azumaya extension of B^H .

Theorem 4.1.

- (1) *If B is an H^* -Galois Azumaya extension of B^H , then B is a commutator H^* -Galois extension of B^H .*
- (2) *If B is an H^* -Galois Hirata extension of B^H equal to B^{H^*} , then B is a commutator H^* -Galois extension of B^H .*

Proof. (1) Let B be an H^* -Galois Azumaya extension of B^H . Then, by Lemma 4.1 in [4], $V_B(B^H)$ is an H^* -Galois extension of $(V_B(B^H))^H$. Since $(V_B(B^H))^H = V_{B^H}(B^H) = Z$, B is a commutator H^* -Galois extension of B^H .

(2) Let B be an H^* -Galois Hirata extension of B^H . Then $V_B(B^H)$ is a left H -Galois extension of C ([8], Theorem 2.6). Hence $V_B(B^H)$ is a finitely generated and projective C -module. Since C is commutative with 1, $V_B(B^H)$ is a progenerator of C . But then the argument used in the proof of Lemma 2.8 in [3] can be applied here. Let $\Omega = \text{Hom}_C(V_B(B^H), V_B(B^H))$. Then, as a left Ω -module, $B \cong V_B(B^H) \otimes_C \text{Hom}_\Omega(V_B(B^H), B)$. Noting that $\text{Hom}_\Omega(V_B(B^H), B) \cong B^{H^*}$ by $f \rightarrow f(1)$ for each f in $\text{Hom}_\Omega(V_B(B^H), B)$, we have that $B \cong V_B(B^H) \otimes_C B^{H^*}$. But $V_B(B^H)$ is a left H -Galois extension of C , so $V_B(V_B(B^H)) = B^H = B^{H^*}$ ([8], Lemma 2.5). Hence $B \cong V_B(B^H) \otimes_C B^H$. Moreover, since $V_B(V_B(B^H)) = B^H$ again, the centers of B^H and B are the same C . Therefore $V_B(B^H)$ is an H^* -Galois extension of C ([3], Lemma 2.8), that is, B is a commutator H^* -Galois extension of B^H .

Next two theorems relate the class of commutator H^* -Galois extensions to the class of H^* -Galois Azumaya extensions and the class of H^* -Galois Hirata extensions.

Theorem 4.2. *Let B be a commutator H^* -Galois extension of B^H . Then B is separable over Z , the center of B^H , if and only if B is an H^* -Galois Azumaya extension*

of B^H .

Proof. (\implies) Since $B \cong B^H \otimes_Z V_B(B^H)$ as a Z -algebra such that $V_B(B^H)$ is an H^* -Galois extension of Z which is a direct summand of $V_B(B^H)$ as a Z -module ([3], Proposition 1.9 and Corollary 1.10). By hypothesis, B is separable over Z , so B^H is separable over Z ([1], Corollary 1.9 on page 45). Thus B^H is an Azumaya Z -algebra. Moreover, since $B \cong B^H \otimes_Z V_B(B^H)$ again, $Z = C^H$. Therefore B is a separable and H^* -Galois extension of B^H which is an Azumaya C^H -algebra; that is, B is an H^* -Galois Azumaya extension of B^H ([4], Theorem 3.4).

(\impliedby) B is an H^* -Galois Azumaya extension of B^H , so B is a separable extension of B^H and B^H is an Azumaya C^H -algebra. Hence B is a separable extension of $Z (= C^H)$ by the transitivity of separable extensions.

Theorem 4.3. *Let B be a commutator H^* -Galois extension of B^H . Then $V_B(B^H)$ is an Azumaya Z -algebra if and only if B is an H^* -Galois Hirata extension of B^H .*

Proof. (\implies) Since $V_B(B^H)$ is an H^* -Galois extension of Z , $B \cong B^H \otimes_Z V_B(B^H)$ as a Z -algebra. But $V_B(B^H)$ is an Azumaya Z -algebra, so B is a Hirata separable extension of B^H . Hence B is an H^* -Galois Hirata extension of B^H .

(\impliedby) Since B is an H^* -Galois extension of B^H , B is a finitely generated and projective B^H -module. By hypothesis, B is a projective Hirata separable extension of B^H and contains B^H as a direct summand as a B^H -bimodule by Theorem 3.1-(2). But then $V_B(B^H)$ is a separable C -algebra and $V_B(V_B(B^H)) = B^H$ ([5], Proposition 1.3 and Proposition 1.4). This implies that the center of B , B^H , and $V_B(B^H)$ are the same $Z (= C = C^H)$. Thus $V_B(B^H)$ is an Azumaya Z -algebra.

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