On Splitting Rings for Azumaya Skew Polynomial Rings

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Abstract. Let B be a ring with 1, ρ an automorphism of B of order n for some integer n, $B[x;\rho]$ the skew polynomial ring in x over B such that $1, x, x^2, \cdots, x^{n-1}$ are independent over B and $x^n \in U(B^\rho)$ where B^ρ is the set of elements in B fixed under ρ and $U(B^\rho)$ are all units in B^ρ, and $\bar{\rho}$ is the inner automorphism of $B[x;\rho]$ induced by x. Assume n is a unit in B. It is shown that for a $\bar{\rho}$-Galois $B[x;\rho]$ over $(B[x;\rho])^{\bar{\rho}}$ or a DeMeyer-Kanzaki Galois B over B^ρ, $B[x;\rho]$ is Azumaya if and only if so is $(B[x;\rho])^{\bar{\rho}}$ or B, and some splitting rings of $B[x;\rho]$, $(B[x;\rho])^{\bar{\rho}}$ and B are coincide.

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1 Introduction

Let B, ρ, $B[x;\rho]$, B^ρ, $\bar{\rho}$, be as in the abstract and C the center of B. We first show that when $B[x;\rho]$ is a $\bar{\rho}$-Galois extension of $(B[x;\rho])^{\bar{\rho}}$ such that $(B[x;\rho])^{\bar{\rho}}$ is a direct summand of $B[x;\rho]$ or when C is a ρ-Galois algebra over C^ρ with Galois group $\langle \rho|_C \rangle$ of order n, then $B[x;\rho]$ is an Azumaya algebra if and only if so is $(B[x;\rho])^{\bar{\rho}}$.

Let A be an Azumaya algebra. It is well known that any separable maximal commutative subalgebra of A is a splitting ring for A ([2, Theorem 5.5, page 64]). We next show that any separable maximal commutative subalgebra of an Azumaya algebra $(B[x;\rho])^{\bar{\rho}}$ is also a splitting ring for $B[x;\rho]$. Moreover, if B is a DeMeyer-Kanzaki Galois extension with Galois group $\langle\rho\rangle$ generated by ρ (that is, B is an Azumaya C-algebra and C is a Galois algebra over C^ρ with Galois group $\langle\rho|_C\rangle \cong \langle\rho\rangle$, see [1]), we obtain the structure of a separable maximal commutative subalgebra F of B and $F\otimes B[x;\rho]$ respectively where $\otimes$ is over C^ρ. The present paper has been revised under the suggestions of the referee. The authors would like to thank the referee for the valuable suggestions.

2 Basic Definitions and Notations

Throughout this paper, B, C, ρ, B^ρ, $B[x;\rho]$, $\bar{\rho}$ will be as in section 1 and Z the center of $B[x;\rho]$. The multiplications in $B[x;\rho]$ are given by $xb=\rho(b)x$ for $b\in B$ and $x^n=v\in U(B^\rho)$. We note that $\bar{\rho}(f)=xfx^{-1}$ for each $f\in B[x;\rho]$, and so $\bar{\rho}$ restricted to B is ρ.

Let A be a subring of a ring B with the same identity 1. We denote $V_B(A)$ the commutator subring of A in B. We call B a separable extension of A if there exist $\{a_i, b_i$ in B, $i=1,2,...,m$ for some integer $m\}$ such that $\sum a_ib_i=1$, and $\sum ba_i\otimes b_i=\sum a_i\otimes b_ib$ for all b in B where $\otimes$ is over A, and a ring B is called a H-separable extension of A if $B\otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B-bimodule. An Azumaya algebra is a separable extension of its center. B is called a ρ-Galois extension of B^ρ if there exist elements $\{c_i, d_i$ in B, $i=1,2,...,m\}$ for some integer m such that $\sum_{i=1}^m c_id_i=1$ and $\sum_{i=1}^m c_i\rho^k(d_i)=0$ for $0<k<n$. The set $\{c_i,d_i\}$ is called a ρ-Galois system for B. B is called a DeMeyer-Kanzaki ρ-Galois extension if B is an Azumaya C-algebra and C is a $\rho|_C$-Galois algebra with $\langle\rho|_C\rangle\cong\langle\rho\rangle$. If B is an Azumaya C-algebra and S is a commutative C-algebra such that $B\otimes_C S\cong \mathrm{Hom}_S(E,E)$ for some S-progenerator E, then S is called a splitting ring for the Azumaya algebra B.

3 Characterizations of Azumaya Skew Polynomial Rings

In this section, we shall give two characterizations of an Azumaya skew polynomial ring $B[x;\rho]$, one in term of $(B[x;\rho])^{\bar{\rho}}$ and another in term of B.

Theorem 3.1. *Let $B[x;\rho]$ be a $\bar{\rho}$-Galois extension of $(B[x;\rho])^{\bar{\rho}}$ which is a direct summand of $B[x;\rho]$ as a bimodule over $(B[x;\rho])^{\bar{\rho}}$. Then, $B[x;\rho]$ is an Azumaya algebra if and only if so is $(B[x;\rho])^{\bar{\rho}}$.*

Proof. Assume $(B[x;\rho])^{\bar\rho}$ is an Azumaya algebra. Since $B[x;\rho]$ is a $\bar\rho$-Galois extension with Galois group $\langle\bar\rho\rangle$ where $\bar\rho$ is the inner automorphism of $B[x;\rho]$ induced by x, that is, $\bar\rho(f)=xfx^{-1}$ for each $f\in B[x;\rho]$. Hence, $B[x;\rho]$ is a H-separable extension of $(B[x;\rho])^{\bar\rho}$ ([7, Corollary 3]). But $(B[x;\rho])^{\bar\rho}$ is an Azumaya algebra, so $B[x;\rho]$ is also an Azumaya algebra ([5, Theorem 1]). Conversely, since $B[x;\rho]$ is a $\bar\rho$-Galois extension of $(B[x;\rho])^{\bar\rho}$, it is a finitely generated and projective left module over $(B[x;\rho])^{\bar\rho}$ ([1, Theorem 1]). But $(B[x;\rho])^{\bar\rho}$ is a direct summand of $B[x;\rho]$ as a bimodule over $(B[x;\rho])^{\bar\rho}$, so $(B[x;\rho])^{\bar\rho}$ is a separable Z-algebra from the fact that $B[x;\rho]$ is an Azumaya algebra over Z by hypothesis (see the proof of Lemma 2, page 120 in [1]). Thus $(B[x;\rho])^{\bar\rho}$ is an Azumaya algebra.

Lemma 3.2. *Let Z be the center of $B[x;\rho]$. If C is a $\rho|_C$-Galois extension of C^ρ with Galois group $\langle\rho|_C\rangle$ of order n, then*

(1) $Z=C^\rho$.

(2) $V_{B[x;\rho]}(C)=B$.

Proof. (1) Since C is a $\rho|_C$-Galois extension of C^ρ with Galois group $\langle\rho|_C\rangle$ of order n, $B[x;\rho]$ is a H-separable extension of B by ([4, Theorem 3.3]). But B is a direct summand of $B[x;\rho]$ as a left B-module, so B satisfies the double centralizer property in $B[x;\rho]$ ([6, Proposition 1.2]), that is, $B=V_{B[x;\rho]}(V_{B[x;\rho]}(B))$. This implies that the center of $B[x;\rho]$ is contained in B, that is, $Z\subseteq B$. Thus, $Z\subseteq C$. For each $z\in Z\subseteq C\subseteq B$, $zx=xz=\rho(z)x$. Hence $\rho(z)=z$ since $\{1,x,x^2,\cdots,x^{n-1}\}$ is a free basis of $B[x;\rho]$ over B. Therefore, $Z\subseteq C^\rho$. Thus, $Z=C^\rho$.

(2) Clearly, $B\subseteq V_{B[x;\rho]}(C)$. Conversely, for each $\sum_{i=0}^{n-1}a_ix^i$ in $V_{B[x;\rho]}(C)$, we have $c(\sum_{i=0}^{n-1}a_ix^i)=(\sum_{i=0}^{n-1}a_ix^i)c$ for each c in C, so $a_i(c-\rho^i(c))=0$ for each i. But C is a $\rho|_C$-Galois extension of C^ρ, so the ideal of C generated by $\{c-\rho^i(c)\mid c\in C\}$ is C ([2, Proposition 1.2-(5), page 80]). Thus $a_i=0$ for each $i>0$. But then $\sum_{i=0}^{n-1}a_ix^i=a_0\in B$. Hence $V_{B[x;\rho]}(C)\subseteq B$, and so $V_{B[x;\rho]}(C)=B$.

Theorem 3.3. *Assume C is a $\rho|_C$-Galois extension of C^ρ with Galois group $\langle\rho|_C\rangle$ of order n, then the following statements are equivalent:*

(1) $B[x;\rho]$ is Azumaya.

(2) B is Azumaya.

(3) $(B[x;\rho])^{\bar\rho}$ is Azumaya.

Proof. (1) $\Longrightarrow$ (2): Assume $B[x;\rho]$ is Azumaya. Since C is a $\rho|_C$-Galois extension of C^ρ, $Z=C^\rho$ by Lemma 3.2-(1). Hence $B[x;\rho]$ is an Azumaya C^ρ-algebra. By Lemma 3.2-(2),

$V_{B[x;\rho]}(C) = B$. Therefore, B is a separable C^ρ-algebra (for C is a separable C^ρ-algebra) by the commutator theroem for Azumaya algebras ([2, Theorem 4.3, page 57]). Hence B is an Azumaya algebra.

(2) $\Longrightarrow$ (1): Since C is a $\rho|_C$-Galois extension of C^ρ with Galois group $\langle \rho|_C \rangle$ of order n, $B[x;\rho]$ is a H-saparable extension of B by ([4, Theorem 3.3]). By hypothesis B is an Azumaya C-algebra, so it is a separable C^ρ-algebra by the transitivity of separable extensions. Thus $B[x;\rho]$ is a separable extension of C^ρ. Moreover, by Lemma 3.2-(1), $Z = C^\rho$, so $B[x;\rho]$ is an Azumaya C^ρ-algebra.

(1) $\Longleftrightarrow$ (3): Since C is a $\rho|_C$-Galois extension of C^ρ with Galois group $\langle \rho|_C \rangle$ of order n, there exists an element $c \in C$ such that $Tr_{\langle \rho \rangle}(c) = 1$ ([2, Corollary 1.3-(1), page 85]). Hence $B[x;\rho]$ is a $\bar{\rho}$-Galois extension of $(B[x;\rho])^{\bar{\rho}}$ with the same Galois system of C such that $(B[x;\rho])^{\bar{\rho}}$ is a direct summand of $B[x;\rho]$ as a bimodule over $(B[x;\rho])^{\bar{\rho}}$ because $0 \longrightarrow (B[x;\rho])^{\bar{\rho}} \longrightarrow B[x;\rho]$ splits through $p: B[x;\rho] \longrightarrow (B[x;\rho])^{\bar{\rho}}$ by $p(f) = Tr_{\langle \bar{\rho} \rangle}(cf)$. Thus (1) $\Longleftrightarrow$ (3) by Theorem 3.1.

4 Splitting Rings

For $B[x;\rho]$ being an Azumaya algebra as given in Theorem 3.1 and Theorem 3.3, we shall show that any separable maximal commutative subalgebra of $(B[x;\rho])^{\bar{\rho}}$ or B is a splitting ring for $B[x;\rho]$. Conversely, some separable maximal commutative subalgebras of $B[x;\rho]$ are shown to be contained in $(B[x;\rho])^{\bar{\rho}}$ (or B) and are also splitting ring for $(B[x;\rho])^{\bar{\rho}}$ (or B). We begin with the well known splitting ring theorem for Azumaya algebras as given in ([2, Theorem 5.5, page 64]).

Proposition 4.1. ([2, Theorem 5.5, page 64]) *Let S be a commutative separable extension of R which is a R-progenerator. Then S splits the Azumaya R-algebra A if and only if there is an Azumaya R-algebra B in [A] in the Brauer group of R such that B contains a maximal commutative subalgebra isomorphic to S.*

Lemma 4.2. *Let $B[x;\rho]$ be a skew polynomial ring, where ρ is an automorphism of B of order n. If $x^n = v \in U(B^\rho)$, then $v \in C^\rho$.*

Proof. For any $b \in B$, $x^n b = \rho^n(b)x^n = bx^n$, so $vb = bv$. Thus $v \in C \cap B^\rho$, that is, $v \in C^\rho$.

Theorem 4.3. *Assume n is a unit in B and $B[x;\rho]$ is an Azumaya algebra over Z. If $B[x;\rho]$ is a $\bar{\rho}$-Galois extension of $(B[x;\rho])^{\bar{\rho}}$, then any separable maximal commutative subalgebra of $(B[x;\rho])^{\bar{\rho}}$ over $\langle x,Z\rangle$ is a splitting ring for $B[x;\rho]$ over Z, where $\langle x,Z\rangle$ is the Z-algebra generated by x.*

Proof. Let F be a separable maximal commutative subalgebra of $(B[x;\rho])^{\bar{\rho}}$ over $\langle x,Z\rangle$. Then $F=V_{(B[x;\rho])^{\bar{\rho}}}(F)$. Hence $F=V_{B[x;\rho]}(F)\cap(B[x;\rho])^{\bar{\rho}}$. Noting that $(B[x;\rho])^{\bar{\rho}}=V_{B[x;\rho]}(\langle x,Z\rangle)$, we have that $V_{B[x;\rho]}(F)\subseteq V_{B[x;\rho]}(\langle x,Z\rangle)=(B[x;\rho])^{\bar{\rho}}$; and so $F=V_{B[x;\rho]}(F)\cap(B[x;\rho])^{\bar{\rho}}=V_{B[x;\rho]}(F)$. This implies that F is a maximal commutative subalgebra of $B[x;\rho]$. Moreover, since n and v are units in B, by Lemma 4.2, $\langle x,Z\rangle\cong\frac{Z[X]}{\langle X^n-v\rangle}$ is a separable Z-algebra ([2, page 113]). Thus F is a separable Z-algebra by the transitivity of separable extensions. Hence F is a splitting ring for $B[x;\rho]$ over Z by Proposition 4.1.

Next is a converse of Theorem 4.3.

Theorem 4.4. *Assume n is a unit in B and $B[x;\rho]$ is an Azumaya algebra over Z. If $B[x;\rho]$ is a $\bar{\rho}$-Galois extension of $(B[x;\rho])^{\bar{\rho}}$, then any separable maximal commutative subalgebra F of $B[x;\rho]$ containing x is contained in $(B[x;\rho])^{\bar{\rho}}$ and is a splitting ring for $(B[x;\rho])^{\bar{\rho}}$.*

Proof. Since $x\in F$, $\langle x,Z\rangle\subseteq F$. Hence $V_{B[x;\rho]}(F)\subseteq V_{B[x;\rho]}(\langle x,Z\rangle)$. By hypothesis, $F=V_{B[x;\rho]}(F)$, so $F\subseteq V_{B[x;\rho]}(\langle x,Z\rangle)\subseteq(B[x;\rho])^{\bar{\rho}}$. This implies that F is a maximal commutative subalgebra of $(B[x;\rho])^{\bar{\rho}}$ since it is a maximal commutative subalgebra of $B[x;\rho]$. Moreover, since F is separable over Z and Z is contained in the center of $(B[x;\rho])^{\bar{\rho}}$. Therefore, F is a separable maximal commutative subalgebra of $(B[x;\rho])^{\bar{\rho}}$; and so it is a splitting ring for Azumaya algebra $(B[x;\rho])^{\bar{\rho}}$ (Proposition 4.1 and Theorem 3.1).

Under the condition as given in Theorem 3.3 that C is a $\rho|_C$-Galois extension of C^ρ with Galois group $\langle\rho|_C\rangle$ of order n, we shall show a relation between the splitting rings for Azumaya algebras $B[x;\rho]$ and B.

Lemma 4.5. *Assume C is a $\rho|_C$-Galois extension of C^ρ with Galois group $\langle\rho|_C\rangle$ of order n which is a unit in B and that $B[x;\rho]$ is an Azumaya algebra. Then the center of $(B[x;\rho])^{\bar{\rho}}$ is $\sum_{i=0}^{n-1}C^\rho x^i$.*

Proof. By Lemma 3.2-(1), the center of $B[x;\rho]$ is $Z = C^\rho$. Hence $B[x;\rho]$ is an C^ρ-Azumaya algebra. Since n and $v(=x^n)$ are units in B, $\sum_{i=0}^{n-1} C^\rho x^i$ is a separable subalgebra of $B[x;\rho]$ ([2, page 113]). Hence the center of $(B[x;\rho])^{\bar\rho}$ is $V_{(B[x;\rho])^{\bar\rho}}((B[x;\rho])^{\bar\rho}) = (B[x;\rho])^{\bar\rho} \cap V_{B[x;\rho]}((B[x;\rho])^{\bar\rho}) = (B[x;\rho])^{\bar\rho} \cap V_{B[x;\rho]}(V_{B[x;\rho]}(\sum_{i=0}^{n-1} C^\rho x^i)) = (B[x;\rho])^{\bar\rho} \cap \sum_{i=0}^{n-1} C^\rho x^i = \sum_{i=0}^{n-1} C^\rho x^i$.

In the following, we shall consider two kinds of separable maximal commutative subalgebras of $B[x;\rho]$, one containing C and the other containing x.

Theorem 4.6. *Assume C is a $\rho|_C$-Galois extension of C^ρ with Galois group $\langle\rho|_C\rangle$ of order n which is a unit in B and that $B[x;\rho]$ is an Azumaya algebra. Then F is a separable maximal commutative subalgebra of $B[x;\rho]$ containing C if and only if F is a separable maximal commutative subalgebra of B.*

Proof. ($\Longrightarrow$) Assume F is a separable maximal commutative subalgebra of $B[x;\rho]$ containing C. Then $C \subseteq F$ and $F = V_{B[x;\rho]}(F)$. Hence $F = V_{B[x;\rho]}(F) \subseteq V_{B[x;\rho]}(C)$. Since C is a $\rho|_C$-Galois extension of C^ρ, $V_{B[x;\rho]}(C) = B$ by Lemma 3.2-(2). Thus $F = V_{B[x;\rho]}(F) \subseteq V_{B[x;\rho]}(C) = B \subseteq B[x;\rho]$; and so F is also a separable maximal commutative subalgebra of B.

($\Longleftarrow$) Let F be a separable maximal commutative subalgebra of B. Then $C \subseteq F$ and $F = V_B(F)$. Hence $V_{B[x;\rho]}(F) \subseteq V_{B[x;\rho]}(C)$. By Lemma 3.2-(2) again, $V_{B[x;\rho]}(C) = B$. Thus $V_{B[x;\rho]}(F) \subseteq V_{B[x;\rho]}(C) = B$. Therefore $V_{B[x;\rho]}(F) = V_B(F)$; and so $F = V_B(F) = V_{B[x;\rho]}(F)$. Hence F is a separable maximal commutative subalgebra of $B[x;\rho]$ containing C.

Theorem 4.7. *Assume C is a $\rho|_C$-Galois extension of C^ρ with Galois group $\langle\rho|_C\rangle$ of order n and that $B[x;\rho]$ is an Azumaya algebra. Let F be a subalgebra of $B[x;\rho]$ containing x. Then,*

(1) F is a maximal commutative subalgebra of $B[x;\rho]$ if and only if F is a maximal commutative subalgebra of $(B[x;\rho])^{\bar\rho}$.

(2) In addition, let n be a unit in B. Then F is a separable subalgebra of $B[x;\rho]$ if and only if F is a separable subalgebra of $(B[x;\rho])^{\bar\rho}$ over $\sum_{i=0}^{n-1} C^\rho x^i$.

Proof. (1) The proof of ($\Longrightarrow$) is in the proof of Theorem 4.4 and the proof of ($\Longleftarrow$) is in the proof of Theorem 4.3.

(2) The proof is in the proof of Theorem 4.4.

For a DeMeyer-Kanzaki Galois extension B, we shall show a structure theorem for a separable maximal commutative subalgebra F of B, $F \otimes B$, and $F \otimes B[x;\rho]$ respectively, where $\otimes$ is over C^ρ.

Theorem 4.8. *Assume B is a DeMeyer-Kanzaki Galois extension. If F is a separable maximal commutative subalgebra of B over C, then*

(1) $F = DC$ where D is a separable maximal commutative subalgebra of B^ρ.

(2) $F \otimes B \cong (Hom_D(B^\rho, B^\rho))^\circ \otimes C$ and $F \otimes B[x;\rho] \cong (Hom_D(B^\rho, B^\rho))^\circ \otimes M_n(C)$, where $\otimes$ is over C^ρ, $M_n(C)$ is the matrix algebra over C of order n, and S° is the opposite algebra of S for an algebra S.

Proof. (1) Since F is a separable subalgebra of B over C, it is a separable C^ρ-algebra (for B is a DeMeyer-Kanzaki Galois extension). Hence F is a separable subalgebra of the Azumaya C^ρ-algebra $B[x;\rho]$ by Theorem 3.3. But $C[x;\rho]$ is also an Azumaya C^ρ-algebra such that $C[x;\rho] \subseteq F[x;\rho]$ (for $C \subseteq F \subseteq B$), so $F[x;\rho] = C[x;\rho] \cdot V_{F[x;\rho]}(C[x;\rho])$ by the commutator theroem for Azumaya algebra $F[x;\rho]$ ([2, Theorem 4.3, page 57]). Since $V_{F[x;\rho]}(C[x;\rho]) = V_{B[x;\rho]}(C[x;\rho]) \cap F[x;\rho] = B^\rho \cap F[x;\rho] = B^\rho \cap F$, $F[x;\rho] = C[x;\rho](B^\rho \cap F)$. Hence $F = C(B^\rho \cap F)$ by comparing the constant terms of both sides. Let $D = B^\rho \cap F$, and noting that F is a maximal commutative subalgebra of B, we have that D is a maximal commutative subalgebra of B^ρ (for $B = B^\rho \cdot C \cong B^\rho \otimes C$ as proved in [1, Lemma 2, page 120]).

Next, we claim that D is a separable subalgebra of B^ρ over C^ρ. In fact, $F = DC \cong D \otimes_{C^\rho} C$ and C^ρ is a direct summand of C ([2, Corollary 1.3, page 85]), so the separability of F over C implies that D is a C^ρ-separable algebra ([2, Corollary 1.10, page 45]).

(2) Since B is a DeMeyer-Kanzaki Galois extension, $B \cong B^\rho \otimes C$ ([1, Lemma 2, page 120]); and so $B[x;\rho] \cong B^\rho \otimes C[x;\rho]$, Thus $F \otimes B \cong (\mathrm{Hom}_D(B^\rho, B^\rho))^\circ \otimes C$ and $F \otimes B[x;\rho] \cong (D \otimes B^\rho) \otimes C \otimes C[x;\rho] \cong (\mathrm{Hom}_D(B^\rho, B^\rho))^\circ \otimes M_n(C)$ where $\otimes$ is over C^ρ ([2, Theorem 5.5, page 64] and [3, Theorem 2.2]).

We conclude the paper with two examples to demonstrate our results.

Example 1 Let Q be the rational field, $B = Q[i,j,k]$ the quaternion algebra over Q, ρ_i the inner automorphism of B induced by i (that is, $\rho_i(b) = ibi^{-1}$ for all $b \in B$), $B[x;\rho_i]$

a skew polynomial ring over B with a free basis $\{1, x \mid x^2 = -1\}$, and $\bar{\rho}_i(f) = xfx^{-1}$ for each $f \in B[x;\rho_i]$. Then,

(1) B is a Galois extension of B^{ρ_i} with Galois group $\{1, \rho_i\}$ of order 2 because $\{\frac{1}{2}, \frac{i}{2}, \frac{j}{2}, \frac{k}{2}\ ;\ \frac{1}{2}, -\frac{i}{2}, -\frac{j}{2}, -\frac{k}{2}\}$ is a Galois system for B with Galois group $\{1, \rho_i\}$.

(2) The center C of B is Q and $\langle \rho_i|_C \rangle = \{1\}$, so C is not $\rho_i|_C$-Galois with Galois group of order 2, that is, B is not a DeMeyer-Kanzaki Galois extension.

(3) Noting that 2 is a unit of B, $B[x;\rho_i]$ is a $\bar{\rho}_i$-Galois extension of $(B[x;\rho_i])^{\bar{\rho}_i}$ which is a direct summand of $B[x;\rho_i]$ as a bimodule over $(B[x;\rho_i])^{\bar{\rho}_i}$.

(4) B is an Azumaya Q-algebra, so $B[x;\rho_i]$ is a separable Q-algebra. Hence $B[x;\rho_i]$ is an Azumaya algebra (for 2^{-1} is in B). Thus $(B[x;\rho_i])^{\bar{\rho}_i}$ is also an Azumaya algebra by (3) and Theorem 3.1.

(5) Since $V_{B[x;\rho_i]}((Q[i])[x]) = (Q[i])[x]$, $(Q[i])[x]$ is a maximal commutative subalgebra of $B[x;\rho_i]$. Noting that $(Q[i])[x]$ is a separable algebra over Q, it is a splitting ring for the Azumaya skew polynomial ring $B[x;\rho_i]$. Hence it is also a splitting ring for $B[x;\rho_i]^{\bar{\rho}_i}$ by Theorem 4.4.

Example 2 Let $M_2(Q)$ be the matrix ring of order 2 over the rational field Q, $B = M_2(Q) \times M_2(Q)$, ρ the twisting automorphism of B (that is, $\rho(a,b) = (b,a)$ for all $(a,b) \in B$), and $B[x;\rho]$ a skew polynomial ring over B with a free basis $\{1, x \mid x^2 = (-I_2, -I_2)\}$ where I_2 is the identity of $M_2(Q)$. Then,

(1) The center C of B is $Q \times Q$ and is a Galois extension of $C^{\rho}(= \{(a,a) \mid a \in Q\})$ with Galois group $\langle \rho|_C \rangle$ of order 2.

(2) B is an Azumaya C-algebra, so it is a DeMeyer-Kanzaki Galois extension by (1). Hence $B[x;\rho]$ satisfies the hypothesis of Theorem 4.6.

(3) Since $V_{M_2(Q)}(Q\left[\begin{matrix}-1 & 0\\ 0 & 1\end{matrix}\right]) = Q\left[\begin{matrix}-1 & 0\\ 0 & 1\end{matrix}\right]$ and $\left[\begin{matrix}-1 & 0\\ 0 & 1\end{matrix}\right]^2 = \left[\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right]$, $Q\left[\begin{matrix}-1 & 0\\ 0 & 1\end{matrix}\right]$ is a separable maximal commutative subslgebra of $M_2(Q)$. Hence $Q\left[\begin{matrix}-1 & 0\\ 0 & 1\end{matrix}\right] \times Q\left[\begin{matrix}-1 & 0\\ 0 & 1\end{matrix}\right]$ is a separable maximal commutative subslgebra of B. Thus it is a splitting ring for B and for $B[x;\rho]$ by Theorem 4.6.

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