Indeterminate Forms and L'Hôpital's Rule

The "rule" that goes by the name of Guillaume François Antoine de L'Hôpital, Marquis de Sainte–Mesme (1651–1704), but which was actually discovered by John Bernoulli, is a technique for evaluating limits of *indeterminate forms*. It is by this rule that Marquis de L'Hôpital is best known today, but his greatest contribution to mathematics may have been as the author of the very first calculus textbook. Though later editions bore his name, the book first appeared anonymously, in 1696, under the somewhat daunting title "*Analyse des infiniments petits, pour l'intelligence des lignes courbes,*" ("*Analysis of the infinitely small, for the study of curves*"). The book was based on (some have said stolen from) notes of John Bernoulli who had learned and molded the calculus after many years of correspondence with Gottfried Leibniz, the man who, together with Isaac Newton, is credited with the discovery of calculus.

What is an indeterminate form? There are various types of such forms, but all come about when, while computing at limit, you're faced with conflicting forces at a certain point. A common example of this occurs when you have a ratio of two terms, each of which is getting arbitrarily close to zero, for example,

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

Here, as x goes to zero, the numerator is getting arbitrarily small, which would normally mean that the whole fraction is getting arbitrarily small; but in this case, the denominator is getting arbitrarily small, too, which would normally mean that the whole fraction is getting arbitrarily big – one force wants to make the fraction big, one small. In essence you are trying to make sense of $\frac{0}{0}$. Who wins? That's what L'Hôpital's rule is designed to answer.

Here's the rule in a form that will answer the question in the example above.

L'Hôpital's Rule, $\frac{0}{0}$. *Let* $f(x)$ *and* $g(x)$ *be functions which are differentiable with continuous derivatives in some deleted interval*[1] *of the point* a. *If*

(a) $f(x)$ *and* $g(x)$ *are both going to zero, as* $x \to a$,

(b) $g'(x)$ *is non–zero in some deleted interval of* a, *and*

(c) $\dfrac{f'(x)}{g'(x)}$ *approaches a limit* L *as* $x \to a$,

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

You can apply this to the example above. There $a = 0$, both the numerator and denominator are differentiable and approach zero as $x \to 0$; also the derivative of the denominator is 1, which is never zero. So, if the ratio of the derivatives approaches a limit, so will the original ratio, and it will be the same limit. The rest is easy;

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1.$$

Example. Evaluate $\lim\limits_{x \to 8} \dfrac{x - 8}{\sqrt[3]{x} - 2}$.
The functions $x - 8$ and $\sqrt[3]{x} - 2$ are both differentiable around 8, and the derivative of the denominator is $\frac{1}{3}x^{-2/3}$ which is only zero when $x = 0$ and so is non-zero in a deleted interval of 8, so L'Hôpital's Rule gives

$$\lim_{x \to 8} \frac{x - 8}{\sqrt[3]{x} - 2} = \lim_{x \to 8} \frac{1}{\frac{1}{3}x^{-2/3}} = 12. \qquad \square$$

The next example is from L'Hôpital's 1696 textbook.

Example. Evaluate $\lim\limits_{x \to 1} \dfrac{(2x - x^4)^{1/2} - x^{1/3}}{1 - x^{3/4}}$.
First check that the limit is an indeterminate form, which it is. Next, check the derivative of the denominator, which is $\frac{3}{4}x^{-1/4}$; it's non–zero in a deleted interval of 1, (which deleted interval?), so you can apply L'Hôpital's Rule as follows

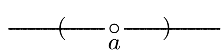
$$\begin{aligned}\lim_{x \to 1} \frac{(2x - x^4)^{1/2} - x^{1/3}}{1 - x^{3/4}} &= \lim_{x \to 1} \frac{\frac{1}{2}(2x - x^4)^{-1/2} \cdot (2 - 4x^3) - \frac{1}{3}x^{-2/3}}{-\frac{3}{4}x^{-1/4}} \\ &= \tfrac{16}{9}. \qquad \square\end{aligned}$$

Now I'll give a proof of Marquis de L'Hôpital's Rule:

I'll start with a couple of preliminary remarks. Since $g'(x)$ is continuous and non-zero, it can't change sign, and so must be always positive or always negative in some deleted interval of a. Next, since $f(x)$ and $g(x)$ are differentiable around a, it does no harm to define, if necessary, $f(a) = g(a) = 0$. I'll also assume that L is some finite real number.

For concreteness, I'll assume that $g'(x)$ is positive and that x approaches a from the right; that is, $a < x$. It'll be easy for you to make the minor modifications needed for the other cases.

[1] [1] A *deleted interval* of the point a, graphically, is of the form



i.e., it's all points, *other* than a, in some open interval containing a.

Now, you can make $f'(x)/g'(x)$ as close to L as you'd care to by making x sufficiently close to a. Algebraically, this means that given any positive number ϵ you can choose x close enough to a so as to make

$$L - \epsilon \leq \frac{f'(x)}{g'(x)} \leq L + \epsilon$$

and, since $g'(x)$ is positive,

$$(L - \epsilon)g'(x) \leq f'(x) \leq (L + \epsilon)g'(x). \tag{1}$$

I'll interject a result which, though obvious, is incredibly useful at various points in the study of calculus.

Race Track Principle. *Suppose the functions $f(x)$ and $g(x)$ are differentiable and $f(c) = g(c)$ at some point c. If $f'(x) \leq g'(x)$, for every $x \geq c$, then $f(x) \leq g(x)$, for every $x \geq c$.*

The reason for this is easy. Think of the function $f(x)$ and $g(x)$ as starting out at the same spot at $x = c$, but from then on $g(x)$ outruns $f(x)$. It must follow that thereafter $g(x)$ is ahead of $f(x)$.

Now apply the Race Track Principle to equation (1). Since $f(a) = 0 = (L + \epsilon)g(a)$, then the right hand side of (1) gives $f(x) \leq (L+\epsilon)g(x)$, for x sufficiently close to a. Similarly $(L - \epsilon)g(a) = 0 = f(a)$, so the left hand side of (1) gives $(L - \epsilon)g(x) \leq f(x)$, for x sufficeintly close to a. Therefore

$$(L - \epsilon)g(x) \leq f(x) \leq (L + \epsilon)g(x). \tag{2}$$

Since $g'(x)$ is positive, $g(x)$ is increasing; as $g(a) = 0$, $g(x)$ is positive for $x > a$. Dividing through equation (2) by the *positive* quantity $g(x)$ you see

$$L - \epsilon \leq \frac{f(x)}{g(x)} \leq L + \epsilon$$

which is equivalent to

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L$$

giving us the desired result. □

The way I stated L'Hôpital's rule above is similar to how M. L'Hôpital originally presented it. But the result has been extended a number of times and is more general.

First, the limit L may be either finite or infinite, meaning that, under the other conditions of the theorem, if $f'(x)/g'(x)$ goes to infinity, so does $f(x)/g(x)$. Second, the point a

may be either finite or infinite [2]. So both the first and third conditions of the theorem allow infinite values. Also the limits may be only one–sided limits.

The most important extension, though, comes from allowing the indeterminate form $\frac{\infty}{\infty}$; that is, allowing the limit of $f(x)$ and $g(x)$ to be infinite. The second condition, however, can't be weakened.

For easy reference, here is L'Hôpital's rule for the indeterminate form $\frac{\infty}{\infty}$.

L'HÔPITAL'S RULE, $\frac{\infty}{\infty}$. *Let $f(x)$ and $g(x)$ be functions which are differentiable with continuous derivatives in some deleted interval of the point a. If*

(a) $f(x)$ and $g(x)$ are both going to $\pm\infty$, as $x \to a$,

(b) $g'(x)$ is non–zero in some deleted interval of a, and

(c) $\dfrac{f'(x)}{g'(x)}$ approaches a limit L as $x \to a$,

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Recall that either a or L may be $\pm\infty$. The proof of this case requires no new ideas beyond those of proof above.

It's not uncommon to have to apply L'Hôpital's Rule more than once before getting your final result. Here's an example of that.

EXAMPLE. Evaluate $\displaystyle\lim_{x\to+\infty} \frac{x^2}{e^x}$.

Here $a = +\infty$ and the limit of both the numerator and denominator is infinite; so you're looking at the indeterminate form $\frac{\infty}{\infty}$. You can check the hypotheses of L'Hôpital's Rule and then compute

$$\lim_{x\to+\infty} \frac{x^2}{e^x} = \lim_{x\to+\infty} \frac{2x}{e^x},$$

which is another indeterminate form. Now repeat the process to get

$$\lim_{x\to+\infty} \frac{2x}{e^x} = \lim_{x\to+\infty} \frac{2}{e^x} = 0. \qquad \square$$

There are other indeterminate forms: $0 \cdot \infty$, 0^0, 1^∞, ∞^0. These can usually be treated by converting them to one of the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

[2] [2] For $a = +\infty$, respectively $-\infty$, a deleted interval of infinity consists of all points greater than, respectively, less than, some finite number.

EXAMPLE. Evaluate $\lim_{x\to 0^+} x\ln x$.
This limit is of the form $0\cdot\infty$. You can convert it to the form $\frac{\infty}{\infty}$ with some simple algebra.

$$\begin{aligned} \lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+} \frac{\ln x}{1/x} \\ &= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} \qquad \text{(L'Hôpital for } \tfrac{\infty}{\infty}\text{)} \\ &= \lim_{x\to 0^+} -x = 0. \end{aligned}$$

□

EXAMPLE. Evaluate $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$.
This limit is of the form 1^∞ . First write

$$\left(1+\frac{1}{x}\right)^x = e^{\ln\left(1+\frac{1}{x}\right)^x} = e^{x\ln\left(1+\frac{1}{x}\right)}$$

and compute $\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)$. This is an indeterminate form $0\cdot\infty$, so you can proceed as in the previous example.

$$\begin{aligned} \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x} \\ &= \lim_{x\to\infty} \frac{-1/(x^2+x)}{-1/x^2} \qquad \text{(L'Hôpital for } \tfrac{\infty}{\infty}\text{)} \\ &= \lim_{x\to\infty} \frac{x^2}{x^2+x} \qquad \text{(L'Hôpital for } \tfrac{\infty}{\infty}\text{)} \\ &= \lim_{x\to\infty} \frac{2x}{2x} = 1. \end{aligned}$$

So

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e^1 = e.$$

□

I want to spend a couple of words on the hypotheses of L'Hôpital's Rule. The first and third conditions are pretty straighforward and it's easy to give examples where the desired conclusion fails if either of them is violated. The second condition, that the derivative be non-zero in some deleted neighborhood of a , is a bit more subtle. At the end here I'll give you an example, not too terribly hard to understand, though tricky to discover, to show that L'Hôpital's Rule can fail if this condition is violated.

EXAMPLE. Let's put

$$f(x) = \tfrac{1}{2}(\sin(x)\cos(x)+x) \qquad g(x) = \tfrac{1}{2}(\sin(x)\cos(x)+x)e^{\sin(x)}.$$

You'll have no trouble checking that

$$f'(x) = \cos^2(x),$$
$$g'(x) = \cos^2(x)e^{\sin(x)} + \tfrac{1}{2}(\sin(x)\cos(x) + x)\cos(x)e^{\sin(x)}.$$

Now, let's look at the limit of the functions and their derivatives as $x \to \infty$. I'll start with

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{e^{\sin(x)}};$$

since $\sin(x)$ oscillates between $+1$ and -1, this ratio oscillates, too, and so the limit does not exist. On the other hand,

$$\begin{aligned}\lim_{x\to\infty} \frac{f'(x)}{g'(x)} &= \lim_{x\to\infty} \frac{\cos^2(x)}{\cos^2(x)e^{\sin(x)} + \frac{1}{2}(\sin(x)\cos(x) + x)\cos(x)e^{\sin(x)}} \\ &= \lim_{x\to\infty} \frac{\cos(x)}{\cos(x)e^{\sin(x)} + \frac{1}{2}(\sin(x)\cos(x) + x)e^{\sin(x)}}.\end{aligned}$$

Note that the functions $\cos(x)$ and $e^{\sin(x)}$ are bounded while $\frac{1}{2}(\sin(x)\cos(x)+x)$ increases without bound, and so this last limit is zero. □

Exercises

Evaluate each of the following limits.

1. $\lim_{x\to 0} \dfrac{x - \sin x}{\tan x}$
2. $\lim_{x\to 0} \dfrac{e^x - 1}{xe^x}$
3. $\lim_{x\to 0} \dfrac{\sqrt{x}}{x^2 - 1}$
4. $\lim_{x\to\infty} xe^{1/x}$
5. $\lim_{x\to 0^+} x^{\sin x}$
6. Explain why the forms $\frac{0}{\infty}, \infty\cdot\infty, 0^\infty$, and 0^∞ are not indeterminate.
7. Define a function $f(x)$ by the formula

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

Sketch a graph of the function (use your TI-92). Use the difference quotient to evaluate $f'(0)$. (Hint: Let $t = 1/x$ and convert the appropriate limit.)